Control of distributed port-Hamiltonian systems

Part2: Boundary control systems

Hector Ramirez, Yann Le Gorrec

FEMTO-ST UMR CNRS 6174, UFC, ENSMM, Besançon, France.

March 2015

Outline

1. Introduction

2. Considered PDEs

3. C_0 Semigroups

4. PHS and generator of C_0 semigroups

5. Boundary control systems

In the first part of this course we focused on properties of Distributed Parameter Systems *i.e.*

$$f = \mathcal{J} e, \; e_\partial, \; f_\partial, \; s.t.$$

$$(f, f_\partial, e, e_\partial) \in \mathcal{D}_\mathcal{J}$$

In Parts 2-3 we focus on solutions associated with the PDE:

- *proving existence of solutions by using the semigroup theory*
- *studying the conditions for asymptotic or exponential stability*
- *designing stabilizing controllers*

Outline

1. Introduction

2. Considered PDEs

3. C_0 Semigroups

4. PHS and generator of C_0 semigroups

5. Boundary control systems

Boundary controlled port Hamiltonian systems

Considered class of PDEs (1D)

$$\frac{\partial x(t,z)}{\partial t} = \mathcal{J}\, \delta_x \mathcal{H}(x(t,z)), \quad \text{with } \mathcal{J} \text{ skew sym. diff. operator}$$

Boundary controlled port Hamiltonian systems

Considered class of PDEs (1D)

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}(\mathcal{L}(z)x)(t,z)) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$$

$P_1 = P_1^\top$, $P_0 = -P_0^\top$, $G_0 \geq 0$, $x \in \mathbb{R}^n$, $z \in (a,b)$, $\mathcal{L}(z) = \mathcal{L}(z)^\top > 0$. State space $X = L_2(a,b;\mathbb{R}^n)$ with $\langle x_1, x_2 \rangle_{\mathcal{L}} = \langle x_1, \mathcal{L}x_2 \rangle$ and the norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1, x_1 \rangle_{\mathcal{L}}$.

Boundary controlled port Hamiltonian systems

Considered class of PDEs (1D)

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}(\mathcal{L}(z)x)(t,z)) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$$

$P_1 = P_1^\top$, $P_0 = -P_0^\top$, $G_0 \geq 0$, $x \in \mathbb{R}^n$, $z \in (a,b)$, $\mathcal{L}(z) = \mathcal{L}(z)^\top > 0$. State space $X = L_2(a,b;\mathbb{R}^n)$ with $\langle x_1, x_2\rangle_{\mathcal{L}} = \langle x_1, \mathcal{L}x_2\rangle$ and the norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1, x_1\rangle_{\mathcal{L}}$.

The norm $\|\cdot\|^2_{\mathcal{L}}$ is equivalent to the energy of the system

Applications

- Mechanical systems, magneto-electro-mechanical, chemical, etc...
- Some beam and wave equations, Maxwell equations, transmission lines, vibrating strings, Saint-Venant equations, ...
- But also by using appropriate extension + closure relations: heat transmission, diffusion systems, tubular reactors, etc...

Boundary controlled port Hamiltonian systems

Considered class of PDEs (1D)

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}(\mathcal{L}(z)x)(t,z)) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$$

$P_1 = P_1^\top$, $P_0 = -P_0^\top$, $G_0 \geq 0$, $x \in \mathbb{R}^n$, $z \in (a,b)$, $\mathcal{L}(z) = \mathcal{L}(z)^\top > 0$. State space $X = L_2(a,b;\mathbb{R}^n)$ with $\langle x_1, x_2 \rangle_\mathcal{L} = \langle x_1, \mathcal{L}x_2 \rangle$ and the norm $\|x_1\|^2_\mathcal{L} = \langle x_1, x_1 \rangle_\mathcal{L}$.

Boundary port variables

Let $\mathcal{L}x \in H^1(a,b;\mathbb{R}^n)$. Then the boundary port variables are the vectors $e_{\partial,\mathcal{L}x}, f_{\partial,\mathcal{L}x} \in \mathbb{R}^n$,

$$\begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} = U \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix} = R \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix}.$$

Where

$$U^T \Sigma U = \Sigma, \quad \Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \Sigma \in M_{2n}(\mathbb{R})$$

C_0 Semi group

In finite dimension, linear systems can be described using first-order differential equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with solutions expressed through:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

The idea of semigroups ([Curtain and Zwart(1995), Jacob and Zwart(2012)]) is to generalize the notion of e^{At} to abstract systems defined on Hilbert space by:

$$\dot{x}(z,t) = Ax(z,t), \ x(z,t) \in D(A), \ x(z,0) = x_0$$

In what follows the semigroup associated to the generator A is noted $T(t)$.

Outline

1. Introduction

2. Considered PDEs

3. C_0 Semigroups

4. PHS and generator of C_0 semigroups

5. Boundary control systems

C_0 Semigroups

Definition of C_0 Semi group

Let X be a Hilbert space. $(T(t))_{t\geq 0}$ is called a strongly continuous semigroup (or C_0) semigroup if the following Holds:

1. For all $t \geq 0$, $T(t)$ is a bounded linear operator on X, *i.e.*, $T(t) \in \mathcal{L}(X)$;
2. $T(0) = 0$;
3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$;
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$ *i.e.* $t \mapsto T(t)$ is strongly continuous at zero.

Even if it has been defined for infinite dimensional systems it can be used in $\mathbb{R}^n$. In this case $T(t) = e^{At}$. Properties can be checked using

$$T(t)x = \Sigma_{n=1}^{\infty} e^{\lambda_n t}\langle x, \phi_n\rangle\phi_n$$

C_0 Semi group

Properties of C_0 Semi group

A strongly continuous semigroup $(T(t))_{t\geq 0}$ on X has the following properties:

1. $\|T(t)\|$ is bounded on every finite sub-interval of $[0, \infty)$;
2. The mapping $t \mapsto T(t)$ is strongly continuous on the interval $[0, \infty)$;
3. For all $x \in X$ we have that $\frac{1}{t}\int_0^t T(s)xds \to x$ as $t \downarrow 0$;
4. If $\omega_0 = inf\left(\frac{1}{t} log\|T(t)\|\right)$ then $\omega_0 = lim\left(\frac{1}{t} log\|T(t)\|\right) < \infty$
5. For every $\omega > \omega_0$, $\exists\ M_\omega$ such that for every $t \geq 0$ we have $\|T(t)\| \leq M_\omega e^{\omega t}$.

Definition of infinitesimal generator

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X. If the following limit exists

$$\lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t} \Rightarrow x_0 \in D(A)$$

we define A the infinitesimal generator of the strongly continuous semigroup by $Ax_0 = \lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t}$

C_0 Semi group

Theorem

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on X with infinitesimal generator A. Then the following results hold:

1. For $x_0 \in D(A)$ and $t \geq 0$ we have $T(t)x_0 \in D(A)$;
2. $\frac{d}{dt}(T(t)x_0) = AT(t)x_0 = T(t)Ax_0$ for $x_0 \in D(A), t \geq 0$;
3. A is a closed linear operator;

It means that for $x_0 \in D(A)$ the function $x(t) = T(t)x_0$ is a solution of the abstract differential equation:

$$\dot{x}(t) = Ax(t),\ x(0) = x_0 \tag{1}$$

Definition

A differentiable function $x : [0, \infty) \to X$ is called classical solution of (1) if $\forall t \geq 0$ we have $x(t) \in D(A)$ and equation (1) is satisfied.

Lemma

Let A be the infinitesimal generator of C_0 semigroup $(T(t))_{t\geq 0}$. Then for every $x_0(t) \in D(A)$ the map $t \mapsto T(t)x_0$ is the unique classical solution of (1).

C_0 Semi group

Definition

Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on X. Then $(T(t))_{t\geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$ and unitary semigroup if $\|T(t)\| = 1$ for every $t \geq 0$.

Definition

A linear operator $A : D(A) \subset X \rightarrow X$ is called dissipative if

$$Re\langle Ax, x\rangle \leq 0, \;\; x \in D(A)$$

Lumer-Phillips Theorem

Let A be a linear operator with domain $D(A)$ on X. Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ on X if and only if A is dissipative and $ran(I - A) = X$

Theorem

Let A be a linear, densely defined and closed operator on X. Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ on X if and only if A and A^* are dissipative.

Outline

1. Introduction

2. Considered PDEs

3. C_0 Semigroups

4. PHS and generator of C_0 semigroups

5. Boundary control systems

Port Hamiltonian systems

Port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W \Sigma W^\top \geq 0$ ($W \Sigma W^\top = 0$), then the operator $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \;\middle|\; \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}$$

generates a contraction semigroup(unitary semigroup) on X.

Sketch of proof

We use the property

$$\langle e, \mathcal{J}e \rangle + \langle \mathcal{J}e, e \rangle = \begin{pmatrix} f_\partial^T & e_\partial^T \end{pmatrix}^T \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$$

to prove that with $D(\mathcal{A})$ with rank and positivity condition the operator and its adjoint are dissipative.

Outline

1. Introduction
2. Considered PDEs
3. C_0 Semigroups
4. PHS and generator of C_0 semigroups
5. Boundary control systems

Boundary control systems

We are interested in abstract control systems of the form:

$$\begin{aligned} \dot{x} &= \mathscr{A}x(t), \quad x(0) = x_0, \\ \mathscr{B}x(t) &= u(t), \end{aligned} \tag{2}$$

Definition

The control system 2 is a boundary control system if the following hold:

1. The operator $A : D(A) \to X$ with $D(A) = D(\mathscr{A}) \cap \ker(\mathscr{B})$ and

   $$\mathscr{A}x = Ax \text{ for } x \in D(A)$$

   is the infinitesimal generator of a C_0 semigroup.
2. There exists an operator $B \in \mathcal{L}(U, X)$ such that for all $u \in U$ we have $Bu \in D(\mathscr{A})$, $\mathscr{A}B \in \mathcal{L}(U, X)$ and

   $$\mathscr{B}Bu = u, \ u \in U$$

Boundary controlled port Hamiltonian systems

Boundary controlled port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W\Sigma W^\top \geq 0$, then the system $\frac{\partial x}{\partial t} = \mathcal{A}x$ with $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \;\middle|\; \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}$$

and input

$$u(t) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}$$

is a Boundary Control System on X.

Sketch of proof

The operator $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain $D(\mathcal{A})$ generates a contraction semigroup on X. It remains to show that $\exists \mathscr{B}$ such that $\mathscr{B}Bu = u, \; u \in U$

Boundary controlled port Hamiltonian systems

Boundary controlled port Hamiltonian systems [Le Gorrec et al.(2005)Le Gorrec, Zwart, and Maschke]

Let $\tilde{W}$ be a full rank matrix of size $n \times 2n$ with $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ invertible and let $P_{W,\tilde{W}}$ be given by

$$P_{W,\tilde{W}} = \left(\begin{bmatrix} W \\ \tilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \tilde{W} \end{bmatrix}^\top \right)^{-1} = \begin{bmatrix} W\Sigma W^\top & W\Sigma\tilde{W}^\top \\ \tilde{W}\Sigma W^\top & \tilde{W}\Sigma\tilde{W}^\top \end{bmatrix}^{-1}.$$

Define the output of the system as the linear mapping $\mathcal{C} : \mathcal{L}^{-1}H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$,

$$y = \mathcal{C}x(t) := \tilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}.$$

Then for $u \in C^2(0,\infty;\mathbb{R}^k)$, $\mathcal{L}x(0) \in H^1(a,b;\mathbb{R}^n)$, and $u(0) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(0) \\ e_{\partial,\mathcal{L}x}(0) \end{bmatrix}$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = \frac{1}{2}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} - \langle G_0\mathcal{L}x(t), \mathcal{L}x(t)\rangle \leq \frac{1}{2}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

Specific cases

Using

$$W = S\,(I + V, I - V)$$
$$\tilde{W} = \tilde{S}\,(I - V, -I - V)$$

We obtain for:

$$V = 0 \begin{cases} \dot{x}(t) &= \mathcal{J}x(t), \\ u(t) &= \frac{1}{2}\,(f_\partial(t) + e_\partial(t)) \\ y(t) &= \frac{1}{2}\,(f_\partial(t) - e_\partial(t)) \end{cases} \implies$$

Scattering system:
Boundary control system with the associated semigroup a contraction
$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2.$

$$V = I \begin{cases} \dot{x}(t) &= \mathcal{J}x(t) \\ u(t) &= f_\partial(t) \\ y(t) &= -e_\partial(t) \end{cases} \implies$$

Impedance passive system
Boundary control system with the associated semigroup unitary
$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$

Back to the vibrating string

The PDE is given by:

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix}\epsilon \\ p\end{pmatrix}}_{f} = \underbrace{\begin{pmatrix}0 & -1 \\ -1 & 0\end{pmatrix}}_{P_1} \frac{\partial}{\partial z} \underbrace{\begin{pmatrix}T(z)\epsilon \\ \frac{1}{\mu(z)}p\end{pmatrix}}_{e}, Q = P_1$$

The boundary port variables are defined by:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}\begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{p(a)}{\mu(a)} - \frac{p(b)}{\mu(b)} \\ T(a)\epsilon(a) - T(b)\epsilon(b) \\ T(a)\epsilon(a) + T(b)\epsilon(b) \\ \frac{p(a)}{\mu(a)} + \frac{p(b)}{\mu(b)} \end{pmatrix}$$

By using the transformation

$$U = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \text{ s.t. } U^T \Sigma U = \Sigma$$

Back to the vibrating string

One can also choose:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} U \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} T(a)\epsilon(a) \\ T(b)\epsilon(b) \\ \frac{p(a)}{\mu(a)} \\ -\frac{p(b)}{\mu(b)} \end{pmatrix}$$

Impedance passive case:

$$V = I \Rightarrow u = \frac{1}{\sqrt{2}} \begin{pmatrix} T(a)\epsilon(a) \\ T(b)\epsilon(b) \end{pmatrix} \text{ and } y = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{p(a)}{\mu(a)} \\ \frac{p(b)}{\mu(b)} \end{pmatrix}$$

$$\frac{dH(t)}{dt} = y(t)^T u(t)$$

Systems with dissipation

Let now consider systems with dissipation:

$$\frac{dx}{dt}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}x(t,z),\ \ x(0,z) = x_0(z),$$

$$\Updownarrow$$

$$\begin{pmatrix} f \\ f_p \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}$$

with $e_p = Sf_p$ where S is a coercive operator

$$\begin{pmatrix} f \\ f_p \end{pmatrix} \in \mathcal{F}, \begin{pmatrix} e \\ e_p \end{pmatrix} \in \mathcal{E} \text{ and } \mathcal{E} = \mathcal{F} = L_2((a,b),\mathbb{R}^n) \times L_2((a,b),\mathbb{R}^n)$$

Systems with dissipation

From geometrical point of view:

$$f_e = \mathcal{J}_e e_e$$

$\mathcal{J}_e$ is formally skew symmetric and can be parametrized by:

$$\mathcal{J}_e \widetilde{e} = \Sigma_1^N \widetilde{P}_k \frac{\partial^k}{\partial z^k} \widetilde{e} \text{ with } \widetilde{P}_k = (-1)^{k+1} \widetilde{P}_k^T$$

In this case $\widetilde{P}_N$ is not full rank and the bilinear product is defined on quotient space.
The extended boundary port variables are defined by:

$$\begin{pmatrix} \widetilde{f}_\partial \\ \widetilde{e}_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{Q}_1 & -\tilde{Q}_1 \\ I & I \end{pmatrix} \begin{pmatrix} M_Q & 0 \\ 0 & M_Q \end{pmatrix} \begin{pmatrix} \widetilde{e}(b) \\ \widetilde{e}(a) \end{pmatrix}$$

M spanning the column of $\widetilde{Q}$, $\widetilde{Q}_1 = M^T \widetilde{Q} M$ and $M_Q = (M^T M)^{-1} M^T$ with

$$\widetilde{Q} = \begin{pmatrix} \widetilde{P}_1 & \widetilde{P}_2 & \cdots & \widetilde{P}_N \\ -\widetilde{P}_2 & -\widetilde{P}_3 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ (-1)^{N-1} \widetilde{P}_N & 0 & \cdots & 0 \end{pmatrix}$$

BCS

Let W be full rank such that $W\Sigma W^T \geq 0$,

$$\frac{dx}{dt}(t) = \mathcal{J}_e x(t)$$

with input

$$u(t) = W \begin{pmatrix} \widetilde{f}_\partial \\ \widetilde{e}_\partial \end{pmatrix}$$

is a **boundary control system**. The operator $A_{\text{ext}} = \mathcal{J}_e$ with domain

$$D(A_{\text{ext}}) = \left\{ \begin{pmatrix} \widetilde{e} \\ \widetilde{e}_r \end{pmatrix} \in \begin{pmatrix} H^N((a,b),\mathbb{R}^n) \\ H^N((a,b),\mathbb{R}^n) \end{pmatrix} \middle| \begin{pmatrix} \widetilde{f}_\partial \\ \widetilde{e}_\partial \end{pmatrix} \in \ker W \right\}, \tag{3}$$

generates a **contraction semigroup**.

Balance equation

Let $\tilde{W}$ be full rank such that $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}$ invertible. Let define $\mathcal{C} : H^N((a,b),\mathbb{R}^{2n}) \to \mathbb{R}^{2nN}$ ast,

$$\mathcal{C}x(t) := \tilde{W} \begin{pmatrix} f_{e,\partial}(t) \\ e_{e,\partial}(t) \end{pmatrix} \tag{4}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{5}$$

then for $u \in C^2((0,\infty);\mathbb{R}^{2nN})$, $x(0) \in H^N((a,b),\mathbb{R}^{2n})$, and $\mathcal{B}x(0) = u(0)$:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{6}$$

where

$$P^{-1}_{W,\tilde{W}} = \begin{pmatrix} W\Sigma W^T & W\Sigma \tilde{W}^T \\ \tilde{W}\Sigma W^T & \tilde{W}\Sigma \tilde{W}^T \end{pmatrix}. \tag{7}$$

Dissipative operator

Now the feedback is closed *i.e.*

$$f = \mathcal{J}e - \mathcal{G}_R S \mathcal{G}_R^* e,$$

The port variables become :

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ (-S\mathcal{G}_R^* e)(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ \frac{d^{N-1}(-S\mathcal{G}_R^* e)}{dz^{N-1}}(b) \\ e(a) \\ (-S\mathcal{G}_R^* e)(a) \\ \vdots \\ \frac{d^{N-1}(-S\mathcal{G}_R^* e)}{dz^{N-1}}(a) \end{pmatrix}, \qquad (8)$$

Dissipative operator

Consider the operator

$$A = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$$

with domain

$$D(A) = \left\{ e \in H^N((a,b);\mathbb{R}^n) \,\middle|\, S\mathcal{G}_R^* e \in H^N((a,b);\mathbb{R}^n), \right. \quad (9)$$

$$\left. \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} \in \ker W \right\}. \quad (10)$$

If W has full rank and satisfies $W \Sigma W^T \geq 0$, then A generates a contraction semigroup.

BCS

Let W be full rank and satisfies $W\Sigma W^T \geq 0$, then

$$\frac{dx}{dt}(t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*) x(t) \tag{11}$$

with input

$$u(t) = \mathcal{B}x(t) = W \begin{pmatrix} g_{f,\partial}(t) \\ g_{e,\partial}(t) \end{pmatrix} \tag{12}$$

is a boundary control system. Furthermore, the operator $A = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ with domain

$$D(A) = \left\{ e \in H^N((a,b);\mathbb{R}^n) \,\middle|\, S\mathcal{G}_R^* e \in H^N((a,b);\mathbb{R}^n), \right. \tag{13}$$

$$\left. \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} \in \ker W \right\}. \tag{14}$$

generates a contraction semigroup.

Dissipative operator

Let $\tilde{W}$ be full rank such that $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}$ invertible. Define the linear mapping $\mathcal{C} : H^N((a,b),\mathbb{R}^{2n}) \to \mathbb{R}^{2nN}$ as,

$$\mathcal{C}x(t) := \tilde{W} \begin{pmatrix} g_{f,\partial}(t) \\ g_{e,\partial}(t) \end{pmatrix} \tag{15}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{16}$$

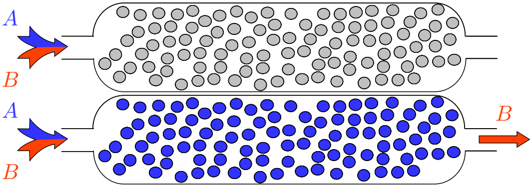
then for $u \in C^2((0,\infty);\mathbb{R}^{2nN})$, $x(0) \in H^N((a,b),\mathbb{R}^{2n})$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 \leq \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{17}$$

where

$$P_{W,\tilde{W}}^{-1} = \begin{pmatrix} W\Sigma W^T & W\Sigma \tilde{W}^T \\ \tilde{W}\Sigma W^T & \tilde{W}\Sigma \tilde{W}^T \end{pmatrix}. \tag{18}$$

Chemical reactor



Let consider a chemical tubular reactor $z \in [a, b]$ with reaction $A \rightarrow B$

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial z}\left(-D_a \frac{\partial C}{\partial z} + vC\right) - k_0 C + \text{Boundary conditions}$$

where $D_a > 0$ and v is a positive constant.

Chemical reactor

Let consider a chemical tubular reactor $z \in [a, b]$ with reaction $A \to B$

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial z}\left(-D_a \frac{\partial C}{\partial z} + vC\right) - kC + \text{Boundary conditions}$$

where $D_a > 0$ and v is a positive constant. By choosing

$$\mathcal{J} = -\frac{\partial}{\partial z}, \ \mathcal{G} = \frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}}, \ \mathcal{G}^* = -\frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}}, \ S = \frac{D_a}{v}$$

$$\begin{pmatrix} \frac{\partial C}{\partial t} \\ f \end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial z} & \frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}} \\ \frac{\partial}{\partial z} - \sqrt{\frac{k}{D_a}} & 0 \end{pmatrix} \begin{pmatrix} vC \\ e \end{pmatrix}$$

Boundary port variables

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} vC(b) \\ D_a \frac{\partial}{\partial z} C(b) - \sqrt{D_a k} C(b) \\ vC(a) \\ D_a \frac{\partial}{\partial z} C(a) - \sqrt{D_a k} C(a) \end{pmatrix}$$

Then

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} vC(a) - vC(b) + D_a \left(\frac{\partial}{\partial z} C(b) - \frac{\partial}{\partial z} C(a) \right) - \sqrt{D_a k} \left(C(b) - C(a) \right) \\ v \left(C(b) - C(a) \right) \\ v \left(C(b) + C(a) \right) \\ D_a \left(\frac{\partial}{\partial z} C(b) + \frac{\partial}{\partial z} C(a) \right) - \sqrt{D_a k} \left(C(b) + C(a) \right) \end{pmatrix}$$

Dankwert conditions

Usually the boundary conditions for tubular reactors are chosen as Dankwert Boundary Conditions:

- input total flow is imposed,
- output dispersive flow is equal to zero.

Dankwert conditions can be written as :

$$vC(t,a) - D_a \frac{\partial C}{\partial z}(t,a) = vC_{\text{in}}(t), \quad \text{and} \quad D_a \frac{\partial C}{\partial z}(t,b) = 0, \tag{19}$$

$$\Leftrightarrow$$

$$\begin{pmatrix} vC_{in} \\ 0 \end{pmatrix} = W \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix}$$

$$W = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & \sqrt{\frac{k_0 D_a}{v}} & 1 - \sqrt{\frac{k_0 D_a}{v}} & -1 \\ 1 & 1 + \sqrt{\frac{k_0 D_a}{v}} & \sqrt{\frac{k_0 D_a}{v}} & 1 \end{pmatrix},$$

Conclusion

One can check that

$$W\Sigma W^T \geq 0$$

iif

$$\sqrt{\frac{kD_a}{v}} \leq \frac{1}{2}$$

It means that he system is a Boundary Control System with associated C_0 semigroup unitary or a contraction if and only if the condition is satisfied.

Otherwise it is not a contraction semigroup.

Outline

1. Introduction
2. Considered PDEs
3. C_0 Semigroups
4. PHS and generator of C_0 semigroups
5. Boundary control systems

Conclusion

In this part we have:

- defined C_0 semigroups, Boundary Control Systems,
- parametrized all the boundary port variables such that the system is a Boundary Control System,
- specified the impedance passive and the scattering cases,
- generalized the result to systems with dissipation

In the next part we will be interested in stability of open/closed loop boundary control systems.

Thank you for your attention !

R.F. Curtain and H.J. Zwart.
An introduction to infinite-dimensional linear systems theory.
Texts in applied mathematics. Springer-Verlag, New York, USA, 1995.

B. Jacob and H.J. Zwart.
Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces, volume 223 of *Operator Theory: Advances and Applications*.
Birkhäuser, Basel, Switzerland, 2012.

Y. Le Gorrec, H. Zwart, and B. Maschke.
Dirac structures and boundary control systems associated with skew-symmetric differential operators.
SIAM Journal on Control and Optimization, 44(5):1864–1892, 2005.