Control of distributed port-Hamiltonian systems

Part3: Stability

Hector Ramirez, Yann Le Gorrec

FEMTO-ST UMR CNRS 6174, UFC, ENSMM, Besançon, France.

March 2015

Outline

Aim of this part

We are now interested in stability of BCS. We consider:

- Asymptotic stability
- Exponential stability

in the case of

- Static boundary feedback
- Dynamic boundary feedback

we will also see how the design dynamic controllers in order to shape the closed loop energy function by using structural invariants.

Stability

We are interested in (exponential) stability of abstract systems of the form

$$\dot{x}(t) = Ax(t), \; t \geq 0, \;\; x(0) = x_0$$

i.e. when the solution tends to zero (exponentially) fast as $t \to 0$.

Definition

The C_0 semigroup $(T(t))_{t\geq 0}$ on X is exponentially stable if there exist positives constants M and α such that

$$\|T(t)\| \leq Me^{-\alpha t} \;\; \textit{for } t \geq 0$$

Stability

Theorem

Suppose that A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ on X. The following are equivalent

1. $(T(t))_{t\geq 0}$ is exponentially stable
2. There exists a positive operator $P \in \mathcal{L}(X)$ such that

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle = -\langle x, x\rangle \text{ for all } x \in D(A) \tag{1}$$

3. There exists a positive operator $P \in \mathcal{L}(X)$ such that

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle \leq -\langle x, x\rangle \text{ for all } x \in D(A)$$

Equation (1) is called Lyapunov equation.

Stability

When there exists a positive operator $P \in \mathcal{L}(X)$ such that

$$\langle Ax, Px \rangle + \langle Px, Ax \rangle \leq 0 \text{ for all } x \in D(A)$$

one has to prove that there exists an invariant set and that this invariant set reduces to zero.
Lassale's invariant principle

Boundary controlled port Hamiltonian systems

Considered class of PDEs (1D)

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}(\mathcal{L}(z)x)(t,z)) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$$

$P_1 = P_1^\top$, $P_0 = -P_0^\top$, $G_0 \geq 0$, $x \in \mathbb{R}^n$, $z \in (a, b)$, $\mathcal{L}(z) = \mathcal{L}(z)^\top > 0$. State space $X = L_2(a, b; \mathbb{R}^n)$ with $\langle x_1, x_2 \rangle_{\mathcal{L}} = \langle x_1, \mathcal{L}x_2 \rangle$ and the norm $\|x_1\|_{\mathcal{L}}^2 = \langle x_1, x_1 \rangle_{\mathcal{L}}$.

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The norm $\|\cdot\|^2_{\mathcal{L}}$ is equivalent to the energy of the system

Boundary controlled port Hamiltonian systems

Considered class of PDEs (1D)

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}\left(\mathcal{L}(z)x)(t,z)\right) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$$

$P_1 = P_1^\top$, $P_0 = -P_0^\top$, $G_0 \geq 0$, $x \in \mathbb{R}^n$, $z \in (a,b)$, $\mathcal{L}(z) = \mathcal{L}(z)^\top > 0$. State space $X = L_2(a,b;\mathbb{R}^n)$ with $\langle x_1, x_2 \rangle_{\mathcal{L}} = \langle x_1, \mathcal{L}x_2 \rangle$ and the norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1, x_1 \rangle_{\mathcal{L}}$.

Boundary port variables

Let $\mathcal{L}x \in H^1(a,b;\mathbb{R}^n)$. Then the boundary port variables are the vectors $e_{\partial,\mathcal{L}x}, f_{\partial,\mathcal{L}x} \in \mathbb{R}^n$,

$$\begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} = U \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix} = R \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix}.$$

Where

$$U^T \Sigma U = \Sigma, \quad \Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \Sigma \in M_{2n}(\mathbb{R})$$

Boundary controlled port Hamiltonian systems

Boundary controlled port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W\Sigma W^\top \geq 0$, then the system $\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}(\mathcal{L}(z)x)(t,z)) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$with input

$$u(t) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}$$

is a BCS on X. The operator $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \;\middle|\; \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}$$

generates a contraction semigroup on X.

Boundary controlled port Hamiltonian systems

Boundary controlled port Hamiltonian systems

Let $\tilde{W}$ be a full rank matrix of size $n \times 2n$ with $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ invertible and let $P_{W,\tilde{W}}$ be given by

$$P_{W,\tilde{W}} = \left(\begin{bmatrix} W \\ \tilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \tilde{W} \end{bmatrix}^\top \right)^{-1} = \begin{bmatrix} W\Sigma W^\top & W\Sigma\tilde{W}^\top \\ \tilde{W}\Sigma W^\top & \tilde{W}\Sigma\tilde{W}^\top \end{bmatrix}^{-1}.$$

Define the output of the system as the linear mapping $\mathcal{C} : \mathcal{L}^{-1}H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$,

$$y = \mathcal{C}x(t) := \tilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}.$$

Then for $u \in C^2(0,\infty;\mathbb{R}^k)$, $\mathcal{L}x(0) \in H^1(a,b;\mathbb{R}^n)$, and $u(0) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(0) \\ e_{\partial,\mathcal{L}x}(0) \end{bmatrix}$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = \frac{1}{2} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} - \langle G_0\mathcal{L}x(t), \mathcal{L}x(t)\rangle \leq \frac{1}{2} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

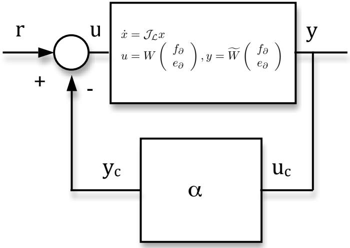
Outline

Closed loop control with static feedback

Impedance passive case

As it has been pointed out in [Villegas(2007)], if the matrices W and $\tilde{W}$ are selected such that $P_{W,\tilde{W}} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \Sigma$, then the BCS fulfills

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} \leq u^{\top}(t)y(t).$$



$$\begin{cases} \dot{x} = \mathcal{J}_{\mathcal{L}} x \\ r = \left(W + \alpha \widetilde{W}\right) \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\ y = \widetilde{W} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \end{cases}$$

Closed loop control with static feedback

Asymptotic stability

Assume that $(\lambda - \mathcal{A})^{-1} : X \to X$ is a compact operator for $\lambda > 0$. Then the system described by:

$$\begin{cases} \dot{x} = \mathcal{J}_{\mathcal{L}} x \\ r = \left(W + \alpha \widetilde{W}\right) \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\ y = \widetilde{W} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \end{cases}$$

with $r = 0$ and $\alpha > 0$ is asymptotically stable.

Sketch of poof

We use the energy as Lyapunov function and Lassale's invariant principle. First the closed loop system is a BCS with infinitesimal generator of a contraction semigroup as soon as $\alpha > 0$. If $u = 0$, $\frac{dH}{dt} = -y^T \alpha y$

Outline

1. Introduction
2. Static feedback
    - Asymptotic stability
    - Exponential stability
3. Dynamic feedback
4. Energy shaping

Exponential stability

Lemma

Consider a BCS such that $W_{cl}\Sigma W_{cl}^T \geq 0$ with $u(t) = 0$, for all $t \geq 0$. Then the energy of the system $E(t) = (1/2)\|x(t)\|^2_{\mathcal{L}}$ satisfies for τ large enough

$$E(\tau) \leq c(\tau)\int_0^\tau \|(\mathcal{L}x)(t,b)\|^2_{\mathbb{R}}dt, \quad \text{and} \quad E(\tau) \leq c(\tau)\int_0^\tau \|(\mathcal{L}x)(t,a)\|^2_{\mathbb{R}}dt,$$

Theorem : exponential stability.

BCS is exponentially stable if the energy of the system satisfies

$$(dE/dt) \leq -k\|(\mathcal{L}x)(t,b)\|^2_{\mathbb{R}} \; or \; (dE/dt) \leq -k\|(\mathcal{L}x)(t,a)\|^2_{\mathbb{R}}$$

where k is a positive real constant.

Example : Timoshenko beam

As state variables we choose

$$
\begin{aligned}
x_1 &= \frac{\partial w}{\partial z} - \phi : && \text{shear displacement,}\\
x_2 &= \rho \frac{\partial w}{\partial t} : && \text{transverse momentum distribution,}\\
x_3 &= \frac{\partial \phi}{\partial z} : && \text{angular displacement,}\\
x_4 &= I_\rho \frac{\partial \phi}{\partial t} : && \text{angular momentum distribution.}
\end{aligned}
$$

Then the model of the beam can be rewritten as

$$
\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P_1} \frac{\partial}{\partial z}\begin{pmatrix} K\,x_1 \\ \frac{1}{\rho}\,x_2 \\ EI\,x_3 \\ \frac{1}{I_\rho}\,x_4 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{P_0} \underbrace{\begin{pmatrix} K\,x_1 \\ \frac{1}{\rho}\,x_2 \\ EI\,x_3 \\ \frac{1}{I_\rho}\,x_4 \end{pmatrix}}_{\mathcal{L}x}.
$$

Velocity feedback

One can define the boundary port variables:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\rho^{-1}x_2)(b) - (\rho^{-1}x_2)(a) \\ (Kx_1)(b) - (Kx_1)(a) \\ (I_\rho^{-1}x_4)(b) - (I_\rho^{-1}x_4)(a) \\ (EIx_3)(b) - (EIx_3)(a) \\ (Kx_1)(b) + (Kx_1)(a) \\ (\rho^{-1}x_2)(b) + (\rho^{-1}x_2)(a) \\ (EIx_3)(b) + (EIx_3)(a) \\ (I_\rho^{-1}x_4)(b) + (I_\rho^{-1}x_4)(a) \end{pmatrix}. \tag{2}$$

Let us consider stabilization by applying velocity feedback *i.e.* following BC:

$$\frac{1}{\rho(a)} x_2(a) = 0, \qquad \frac{1}{I_\rho(a)} x_4(a) = 0,$$

$$K(b)x_1(b,t) = -\alpha_1 \frac{1}{\rho(b)} x_2(b,t), \quad EI(b)x_3(b,t) = -\alpha_2 \frac{1}{I_\rho(b)} x_4(b)$$

Velocity feedback

Input mapping:

$$W_{cl} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ \alpha_1 & 1 & 0 & 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 & 0 & 0 & 1 & \alpha_2 \end{bmatrix}$$

then

$$W_{cl} \Sigma W_{cl}^T = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} \geq 0$$

As output we can choose

$$y = \begin{pmatrix} -K(a)x_1(a) \\ -(EI)(a)x_3(a) \\ \frac{1}{\rho(b)} x_2(b) \\ \frac{1}{I_{\rho(b)}} x_4(b) \end{pmatrix}, \quad \text{with} \quad \widetilde{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Velocity feedback

Then

$$P_{W,\tilde{W}}^{-1} = \begin{bmatrix} 2\alpha & I \\ I & 0 \end{bmatrix}, P_{W,\tilde{W}} = \begin{bmatrix} 0 & I \\ I & -2\alpha \end{bmatrix}$$

Energy balance:

$$\frac{d}{dt}E(t) = \frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_{\mathbb{R}}$$

where

$$\langle \alpha y(t), y(t)\rangle_{\mathbb{R}} = \alpha_1|(\rho^{-1}x_2)(b,t)|^2 + \alpha_2|(I^{-1}x_4)(b,t)|^2$$

Then

$$\begin{aligned} \| (\mathcal{L}x(b)) \|_{\mathbb{R}}^2 &= |(kx_1)(b)|^2 + |(\rho^{-1}x_2)(b)|^2 + |(EIx_3)(b)|^2 + |(I_\rho^{-1}x_4)(b)|^2 \\ &= (\alpha_1^2+1)|(\rho^{-1}x_2)(b,t)|^2 + (\alpha_2^2+1)|(I_\rho^{-1}x_4)(b)|^2 \\ &\leq \kappa\langle \alpha y(t), y(t)\rangle_{\mathbb{R}} = -\kappa\frac{d}{dt}E(t) \end{aligned}$$

$\Rightarrow$ Stability

Dynamic boundary feedback

Consider a strictly passive linear finite dimensional system

$$\dot{v} = A_c v + B_c u_c, \qquad y_c = C_c v + D_c u_c.$$

with storage function $E_c(t) = \frac{1}{2}\langle v(t) Q_c v(t)\rangle_{\mathbb{R}^m}$, $Q_c = Q_c^\top > 0 \in \mathbb{R}^m \times \mathbb{R}^m$.

Theorem [Villegas(2007)]

Let the open-loop BCS satisfy $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = u(t)y(t)$. Consider a LTI strictly passive finite dimensional system with storage function $E_c(t) = \frac{1}{2}\langle v(t), Q_c v(t)\rangle_{\mathbb{R}^m}$. Then the power preserving feedback interconnection

$$u = r - y_c, \qquad\qquad y = u_c,$$

with $r \in \mathbb{R}^n$ the new input of the system is a BCS on the extended state space $\tilde{x} \in \tilde{X} = X \times V$ with inner product $\langle \tilde{x}_1, \tilde{x}_2\rangle_{\tilde{X}} = \langle x_1, x_2\rangle_{\mathcal{L}} + \langle v_1, Q_c v_2\rangle_V$. Furthermore, the operator $\mathcal{A}_e$ defined by

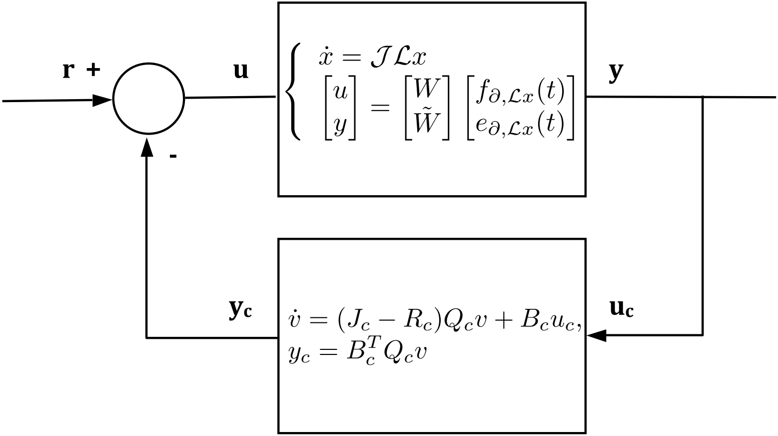
$$\mathcal{A}_e \tilde{x} = \begin{bmatrix} \mathcal{J}\mathcal{L} & 0 \\ B_c \mathcal{C} & A_c \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}, \quad D(\mathcal{A}_e) = \left\{ \begin{bmatrix} x \\ v \end{bmatrix} \in \begin{bmatrix} X \\ V \end{bmatrix} \middle| \mathcal{L}x \in H^N(a, b; \mathbb{R}^n), \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \\ v \end{bmatrix} \in \ker \tilde{W}_D \right\}$$

where

$$\tilde{W}_D = \begin{bmatrix} (W + D_c \tilde{W} & C_c) \end{bmatrix}$$

generates a contraction semigroup on $\tilde{X}$.

Dynamic boundary feedback



Asymptotic stability

Finite dimensional port Hamiltonian controller

$$\dot{v} = (J_c - R_c)Q_c v + B_c u_c, \quad y_c = B_c^\top Q_c v, \quad E_c(t) = \frac{1}{2} v(t)^\top Q_c v(t)$$

where we assume that $Q_c = Q_c^\top > 0$, $J_c = -J_c^\top$, $R_c = R_c^\top \geq 0$ and B_c are real constant matrices of proper dimensions. Furthermore, the controller is assumed to be exponentially stable, i.e., $A_c := (J_c - R_c)Q_c$ is Hurwitz.

Theorem

Consider the above controller connected to the impedance passive system through $u = r - y_c, u_c = y$. Then the operator $\mathcal{A}_e$ described in the previous theorem has compact resolvant.

Theorem

Consider the feedback system $u = r - y_c, u_c = y$ where the controller is chosen satisfying the condition above. Then the closed loop system such that $r = 0$ is globally asymptotically stable.

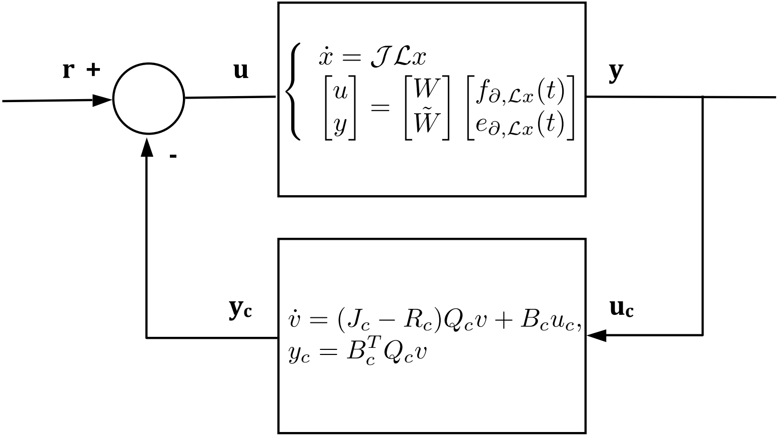
Sketch of proof

- Let first consider that $\omega(0) \in D(\mathcal{A}_e)$. By the aforementioned Theorem [Villegas(2007)], $\mathcal{A}_e$ generates a contraction semigroup.
- Let now consider the energy as Lyapunov function $E_c(t) = \frac{1}{2}\langle \omega(t), \omega(t)\rangle_{\tilde{X}}$. Since $\omega(0) \in D(\mathcal{A}_e)$ and:

$$\frac{dE_c(t)}{dt} = \langle \dot{\omega}(t), \omega(t)\rangle_{\tilde{X}} = \langle \mathcal{A}_e\omega(t), \omega(t)\rangle_{\tilde{X}} = -v^T Q_d v \tag{3}$$

  where $Q_d > 0$. Since $(\lambda I - \mathcal{A}_e)^{-1}$ is compact and the semigroup is a contraction it follows from LaSalle's invariance principle that all solutions asymptotically tend to the maximal invariant set $\mathcal{O}_c = \left\{\tilde{x} \in \tilde{X} | \dot{E}_c = 0\right\}$.
- Let $\mathcal{E}$ be the largest invariant subset of $\mathcal{O}_c$. We can prove that $\mathcal{E} = \{0\}$. From $\dot{E}_c(t) = 0$ and (3) we have $v(t) = 0$ and then $\dot{v}(t) = 0$. Let $\eta < n$ be the rank of $\ker(B_c)$. Form the controller structure $y_c = 0$ and $n - \eta > 0$ components of u_c equal 0. It follows that $\mathcal{O}_c$ reduces to the solution of a first order PDE of dimension n with $2n - \eta$ boundary variables set to zero. It follows from Holmgren's Theorem that $\tilde{x}(t) = 0$, hence the asymptotic stability. The same hold for $\omega(0) \in \tilde{X}$ by using denseness argument [**?**].

Dynamic boundary feedback



Control through energy shaping

Idea:

Use the total energy as Lyapunov function candidate

From the power preserving interconnection:

$$\widetilde{E}(x, v) = E(x) + E_c(v)$$

We are looking for Casimir functions (structural invariants $\Rightarrow \dot{C} = 0$) on the form:

$$C(x, v) = v - F(x)$$

then

$$v - F(x) = \kappa$$

And

$$\widetilde{E}(x, v) = E(x) + E_c(F(x) + \kappa)$$

It remains to choose E_c and to add dissipation such that:

$$\frac{\partial \widetilde{E}}{\partial x}(x^*) = 0, \text{ and } \frac{dE}{dt}(x) < 0$$

Casimir

Let consider the structural invariants of the closed loop system *i.e.* Casimirs, of the form:

$$C(x(t), v(t)) = \Gamma^\top v(t) + \int_a^b \Psi^\top(z) x(t,z) dz \tag{4}$$

with $\Gamma \in \mathbb{R}^m$, $\Psi(z) \in \mathbb{R}^n$ and $\Psi^\top(z)x(t,z) \in H^1(a,b;\mathbb{R}^n)$.

Computation of Casimir functions

Let consider the previously defined boundary controlled port Hamiltonian system with $r = 0$. Then (8) is a Casimir function for the closed loop system if and only if:

$$P_1 \frac{\partial}{\partial z}\Psi(z) + (P_0 + G_0)\Psi(z) = 0, \tag{5}$$

$$(J_c + R_c)\Gamma + B_c \tilde{W} R \begin{bmatrix} \Psi(b) \\ \Psi(a) \end{bmatrix} = 0, \tag{6}$$

$$B_c^\top \Gamma + WR \begin{bmatrix} \Psi(b) \\ \Psi(a) \end{bmatrix} = 0. \tag{7}$$

Energy shaping

From the power preserving interconnexion:

$$\tilde{E}(x, v) = E(x) + E_c(v)$$

We are looking for Casimirs on the form:

$$C(x, v) = v + F(x)$$

then

$$v + F(x) = \kappa$$

And

$$\tilde{E}(x, v) = \tilde{E}(x) = E(x) + E_c(-F(x) + \kappa)$$

It remains to choose E_c s.t.

$$\frac{\partial \tilde{E}}{\partial x}(x^*) = 0$$

Casimirs

Let consider the structural invariants of the closed loop system *i.e.* Casimirs, of the form:

$$C(x(t), v(t)) = \Gamma^\top v(t) + \int_a^b \Psi^\top(z) x(t,z) dz \tag{8}$$

with $\Gamma \in \mathbb{R}^m$, $\Psi(z) \in \mathbb{R}^n$ and $\Psi^\top(z)x(t,z) \in H^1(a,b;\mathbb{R}^n)$.

Casimirs

Consider the previously defined BCS with $r = 0$, where the controller is a dissipative port Hamiltonian controller. Then (8) is a Casimir function for the extended system if:

$$P_1 \frac{\partial}{\partial z}\Psi(z) + (P_0 + G_0)\Psi(z) = 0, \tag{9}$$

$$(J_c + R_c)\Gamma + B_c \tilde{W} R \begin{bmatrix} \Psi(b) \\ \Psi(a) \end{bmatrix} = 0, \tag{10}$$

$$B_c^\top \Gamma + W R \begin{bmatrix} \Psi(b) \\ \Psi(a) \end{bmatrix} = 0. \tag{11}$$

Exponential stability: result

Theorem

Consider the BCS previously defined with $r(t) = 0$, for all $t \geq 0$. It is exponentially stable as soon as

- the finite dimensional boundary controller is exponentially stable and strictly input passive
- and $\|u(t)\|^2 + \|y(t)\|^2 \geq \epsilon \|\mathcal{L}x(t,b)\|^2, \ \epsilon > 0$

- The proof follows the same steps as in [Villegas et al.(2009)Villegas, Zwart, Le Gorrec, and Maschke] including the energy contribution of the finite dimensional controller.
- We used the contraction properties of the total energy used as Lyapunov function, the condition on the interconnection and the exponential stability of the controller.

Exponential stability: assumptions

Finite dimensional port Hamiltonian controller

$$\dot{v} = (J_c - R_c)Q_c v + B_c u_c, \quad y_c = B_c^\top Q_c v + S_c u_c, \quad E_c(t) = \frac{1}{2}v(t)^\top Q_c v(t)$$

where we assume that $Q_c = Q_c^\top > 0$, $J_c = -J_c^\top$, $R_c = R_c^\top \geq 0$, $S_c = S_c^\top > 0$ and B_c are real constant matrices of proper dimensions. Furthermore, the controller is assumed to be exponentially stable, i.e., $A_c := (J_c - R_c)Q_c$ is Hurwitz.
The system is a strictly input passive port-Hamiltonian system, i.e. there exists a $\sigma > 0$ such that

$$\dot{E}_c(t) \leq u_c(t)^\top y_c(t) - \sigma \|u_c(t)\|^2.$$

Input and output of the BCS

We also assume that the BCS satisfies

$$\|u(t)\|^2 + \|y(t)\|^2 \geq \epsilon \|\mathcal{L}x(t,b)\|^2 \tag{12}$$

for some $\epsilon > 0$.

Exponential stability : existence of solutions

Theorem [Villegas(2007)]

Let the open-loop BCS satisfy $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = u(t)y(t)$. Consider a LTI strictly passive finite dimensional system with storage function $E_c(t) = \frac{1}{2}\langle v(t), Q_c v(t)\rangle_{\mathbb{R}^m}$. Then the power preserving feedback interconnection

$$u = r - y_c, \qquad\qquad y = u_c,$$

with $r \in \mathbb{R}^n$ the new input of the system is a BCS on the extended state space $\tilde{x} \in \tilde{X} = X \times V$ with inner product $\langle \tilde{x}_1, \tilde{x}_2\rangle_{\tilde{X}} = \langle x_1, x_2\rangle_{\mathcal{L}} + \langle v_1, Q_c v_2\rangle_V$. Furthermore, the operator $\mathcal{A}_e$ defined by

$$\mathcal{A}_e\tilde{x} = \begin{bmatrix} \mathcal{J}\mathcal{L} & 0 \\ B_c\mathcal{C} & A_c \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}, \quad D(\mathcal{A}_e) = \left\{ \begin{bmatrix} x \\ v \end{bmatrix} \in \begin{bmatrix} X \\ V \end{bmatrix} \middle| \mathcal{L}x \in H^N(a, b; \mathbb{R}^n), \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \\ v \end{bmatrix} \in \ker \tilde{W}_D \right\}$$

where

$$\tilde{W}_D = \begin{bmatrix} (W + D_c\tilde{W} & C_c) \end{bmatrix}$$

generates a contraction semigroup on $\tilde{X}$.

Proof (1)

Idea: use $\tilde{E} = E(x) + E_c(v)$ as Lyapunov function

Lemma

Consider the controlled BCS with $r(t) = 0$, for all $t \geq 0$. Due to the contraction property the energy of the system $\tilde{E}(t) = \frac{1}{2}\|x(t)\|^2_{\mathcal{L}} + \frac{1}{2}v(t)^T Q_c v(t)$ satisfies for τ large enough

$$\tilde{E}(\tau) \leq c(\tau) \int_0^{\tau} \|(\mathcal{L}x)(t,b)\|^2 dt + \frac{2c(\tau)}{c_1} \int_0^{\tau} E_c(t)dt,$$
$$\tilde{E}(\tau) \leq c(\tau) \int_0^{\tau} \|(\mathcal{L}x)(t,a)\|^2 dt + \frac{2c(\tau)}{c_1} \int_0^{\tau} E_c(t)dt,$$

where c is a positive constant that only depends on τ and c_1 a positive constant.

Proof (1)

In order to prove the exponential stability we need the following lemmas

Lemma

There exist strictly positive constants κ_2, κ_3 and κ_4 such that for all $\tau > 0$ the energy of the PH controller satisfies:

$$E_c(\tau) \leq \kappa_1(\tau) E_c(0) + \kappa_3 \int_0^\tau \|u_c(t)\|^2 dt \tag{13}$$

where $\kappa_1(\tau) = \kappa_4 e^{-\kappa_2 \tau}$.

Lemma

There exists positive constants ξ_1, ξ_2 and τ_0 such for all $\tau > \tau_0$ the energy of the PH controller satisfies

$$\int_0^\tau E_c(t) dt \leq \xi_1 \int_0^\tau v^\top(t) Q_c R_c Q_c v(t) dt + \xi_2 \int_0^\tau \|u_c(t)\|^2 dt$$

Lemma

For every $\delta_1 > 0$ there exists a $\delta_2 > 0$ such that for all $\tau > 0$ the energy of the PH controller satisfies the relation

$$\int_0^\tau \delta_1 E_c(t) + \|y_c(t)\|^2 dt \leq \delta_2 \int_0^\tau E_c(t) + \|u_c(t)\|^2 dt. \tag{14}$$

Proof (2)

Let $\sigma > 0$ be such that $S_c \geq \sigma I$. The time derivative of the total energy satisfies

$$
\begin{aligned}
\dot{\tilde{E}} &= -v^\top Q_c R_c Q_c v - u_c^\top S_c u_c \\
&\leq -v^\top Q_c R_c Q_c v - \sigma u_c^\top u_c, \qquad \text{since } S_c \geq \sigma I \\
&= -v^\top Q_c R_c Q_c v - \sigma\epsilon_1 u_c^\top u_c - \sigma\epsilon_2 u_c^\top u_c \\
&= -v^\top Q_c R_c Q_c v - \sigma\epsilon_1 \|u_c\|^2 - \sigma\epsilon_2 \left(\|y\|^2 + \|u\|^2\right) + \\
&\quad \sigma\epsilon_2 \|u\|^2
\end{aligned}
$$

with $\epsilon_1 + \epsilon_2 = 1$ and where we have used that $u_c = -y$.

Proof (3)

Using our main Assumption we have

$$\dot{\tilde{E}} \quad \leq \quad -v^\top Q_c R_c Q_c v \quad - \quad \sigma\epsilon_1 \|u_c\|^2 \quad - \quad \sigma\epsilon_2\epsilon\|\mathcal{L}x(t,b)\|^2 \quad + \quad \sigma\epsilon_2\|y_c\|^2.$$

Integrating this equation on $t \in [0,\tau]$ we have

$$\tilde{E}(\tau) - \tilde{E}(0) \leq -\int_0^\tau v^\top(t) Q_c R_c Q_c v(t)dt$$
$$+\int_0^\tau -\sigma\epsilon_1\|u_c(t)\|^2 - \sigma\epsilon_2\epsilon\|\mathcal{L}x(t,b)\|^2 + \sigma\epsilon_2\|y_c(t)\|^2 dt.$$

Next choose τ sufficiently large such that Lemmas 2 and 3 hold. Using the latter lemma we have

$$\tilde{E}(\tau) - \tilde{E}(0) \leq -\int_0^\tau v^\top Q_c R_c Q_c v + \sigma\epsilon_1\|u_c\|^2 dt$$
$$+\frac{\sigma\epsilon_2\epsilon}{c(\tau)}\left(\frac{2c(\tau)}{c_1}\int_0^\tau E_c(t)dt - \tilde{E}(\tau)\right) + \sigma\epsilon_2\int_0^\tau \|y_c\|^2 dt.$$

Grouping terms we have that

$$\tilde{E}(\tau)\left(1+\frac{\sigma\epsilon_2\epsilon}{c(\tau)}\right) - \tilde{E}(0) \leq$$
$$-\int_0^\tau v(t)^\top Q_c R_c Q_c v(t)dt - \sigma\epsilon_1\int_0^\tau \|u_c(t)\|^2 dt$$
$$+\sigma\epsilon_2\left(\int_0^\tau \frac{2\epsilon}{c_1}E_c(t) + \|y_c(t)\|^2 dt\right).$$

Proof (4)

Using Lemma 3 with $\delta_1 = \frac{2\epsilon}{c_1}$ we have

$$\tilde{E}(\tau)\left(1+\frac{\sigma\epsilon_2\epsilon}{c(\tau)}\right)-\tilde{E}(0)\leq-\int_0^\tau v(t)^\top Q_c R_c Q_c v(t)dt + \sigma\epsilon_2\delta_2\int_0^\tau E_c(t)dt+\sigma(\epsilon_2\delta_2-\epsilon_1)\int_0^\tau \|u_c(t)\|^2 dt. \quad (15)$$

Now, using Lemma 2 we obtain

$$\tilde{E}(\tau)\left(1+\frac{\sigma\epsilon_2\epsilon}{c(\tau)}\right)-\tilde{E}(0)\leq (\sigma\epsilon_2\delta_2\xi_1-1)\int_0^\tau v(t)^\top Q_c R_c Q_c v(t)dt + \sigma(\epsilon_2\delta_2(1+\xi_2)-\epsilon_1)\int_0^\tau \|u_c(t)\|^2 dt.$$

Since ϵ_2 may be chosen to be arbitrarily small, i.e, $\epsilon_2 \ll 1$ and since $\epsilon_1 = 1-\epsilon_2$, we finally have that $\tilde{E}(\tau)\leq c_2\tilde{E}(0)$ with $c_2 = \frac{1}{\left(1+\frac{\sigma\epsilon_2\epsilon}{c(\tau)}\right)} < 1$ which proves the theorem. [**?**]

Outline

Conclusion

In this part we where interested in asymptotic and exponential stability of boundary controlled port Hamiltonian systems. In the static case we gave:

- necessary and sufficient condition on the feedback such that the system is asymptotically stable,
- sufficient condition for exponential stability.

In the dynamic case:

- we have shown that the controller has to be exponentially stable to have asymptotic stability.
- we gave a parametrization with associate conditions of the Casimir functions,
- we have shown that the controller has to be exponentially stable and strictly input passive to have exponential stability.

Thank you for your attention !

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