Introduction to modeling of Port Hamiltonian Systems

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Outline

1. Dynamic systems modeling

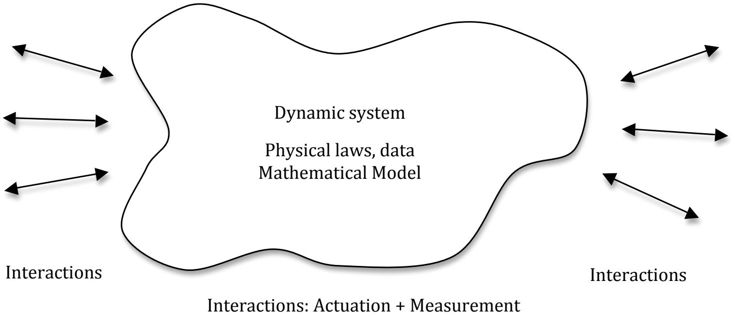
2. Port Hamiltonian framework

3. Back to modeling

4. Port Hamiltonian modeling

Dynamic systems

Modeling of (deterministic) dynamic systems



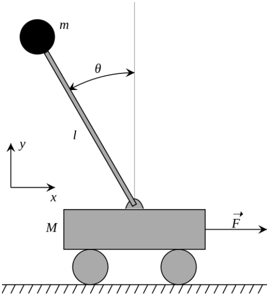
In this course :

Non linear, multi physic, multi scale, distributed parameters systems.

Example 1 : inverted pendulum system

Example : Segway system

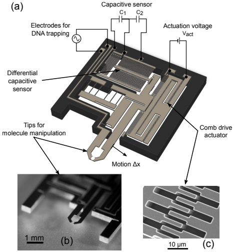
Example 1 : inverted pendulum system

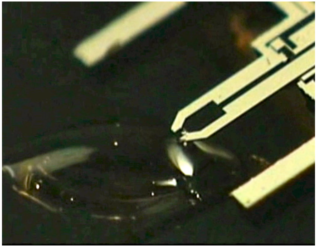


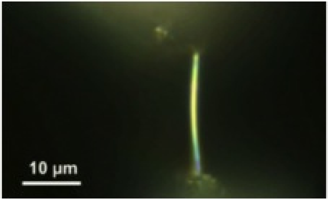
Non linear mechanical system :

- Two natural equilibria.
- Control : insure $\Theta = 0$

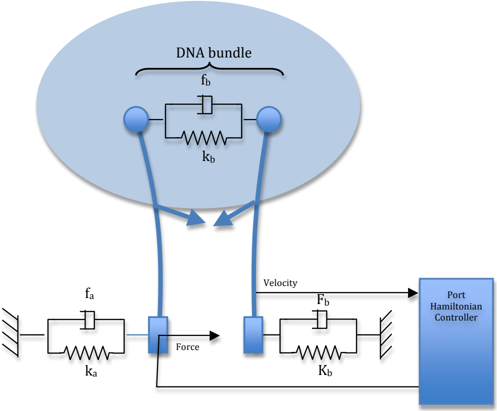
Example 2 : Nanotweezer for DNA manipulation







Example 2 : Nanotweezer for DNA manipulation



Example 3 : Ionic Polymer Metal Composite

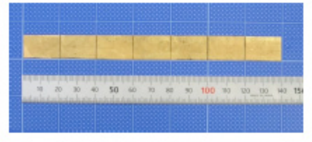
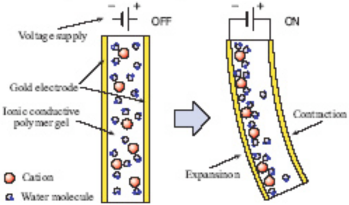
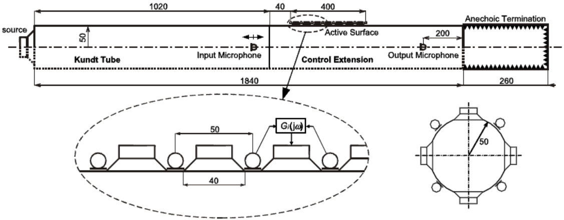


Figure 3. Beam-shaped IPMC actuator



- Electromechanical system.
- 3 scales : Polymer-electrode interface, diffusion in the polymer, beam bending.

Example 4 : Active skin for vibro-acoustic control

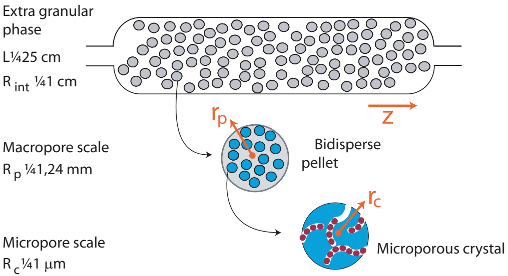


$$\frac{d}{dt}\begin{bmatrix}\theta \\ \Gamma\end{bmatrix} = \begin{bmatrix} 0 & -\overrightarrow{\text{grad}} \\ -\text{div} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\rho_0} & 0 \\ 0 & \frac{1}{\chi_s} \end{bmatrix} \begin{bmatrix}\theta \\ \Gamma\end{bmatrix}$$

2-D case :

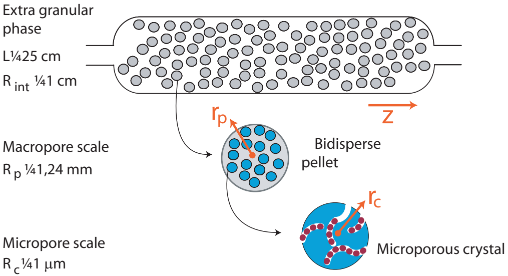
- 2-D wave equation
- Non linear finite dimensional system : loudspeakers/microphones
- Power preserving interconnection

Example 5 : Adsorption process



- Multiscale heterogeneous system.
- Dynamic behavior driven by irreversible thermodynamic laws

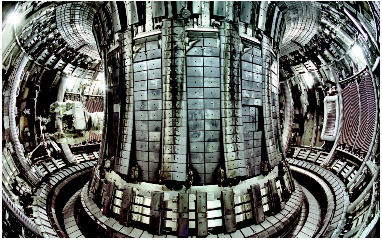
Example 5 : Adsorption process

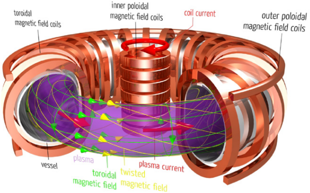


- Multiscale heterogeneous system.
- Considered phenomena :
  - Fluid scale : convection, dispersion.
  - Pellet scale : diffusion (Stephan-Maxwell).
  - Microscopic scale : Knudsen law.

Toward more complex systems ...

Tokamak nuclear reactor





Models and Complexity

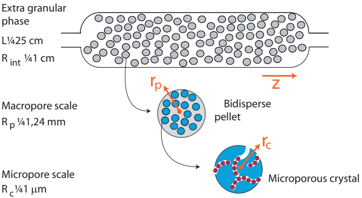
- A model is always an approximation of reality.
- A model depends on the problem context.
- A model has to be tractable.

Purpose

Derive a mathematical model based on Physics useful for :

- Simulation (model reduction)
- Analysis
- Control design

Models and Complexity (illustration)



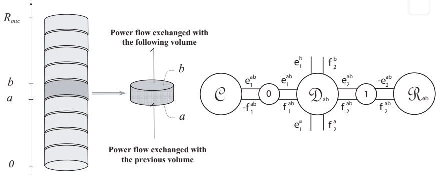
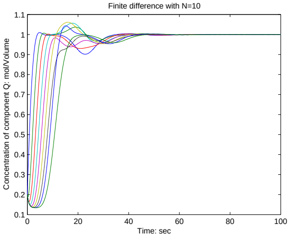
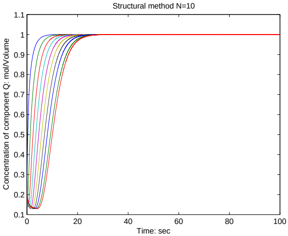


Figure 4: Principle of the spatial discretization





Outline

1. Dynamic systems modeling

2. Port Hamiltonian framework

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Port Hamiltonian framework

Philosophy

- Geometric framework based on a universal conserved quantity : the Energy.
- Use of power conjugated variables names "flows" and "efforts" variables.
- Associated with a powerful graphical tool : the Bond Graphs.

Characteristics

- Formalism coming from differential geometry (free of coordinates, useful for model reduction).
- Suitable for functional analysis (finite and infinite dimension) and system control theory.

Port Hamiltonian framework

Port Hamiltonian systems

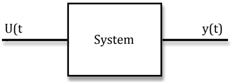
Class of non linear dynamic systems derived from an extension to open physical systems (1992) of Hamiltonian and Gradient systems. This class has been generalized (2001) to distributed parameter systems.

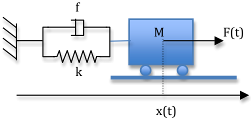
$$x(t) : \begin{cases} \dot{x} = (J(x) - R(x))\frac{\partial H(x)}{\partial x} + B(x)u \\ y = B(x)^T \frac{\partial H(x)}{\partial x} \end{cases} \qquad x(t,z) : \begin{cases} \dot{x} = (\mathcal{J}(x) - \mathcal{R}(x))\frac{\delta \mathcal{H}(x)}{\delta x} \\ \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{\delta \mathcal{H}(x)}{\delta x}\Big|_\partial \end{cases}$$

- Central role of the energy.
- Additional information coming from the geometric structure.
- Multi-physic framework.

A simple example ...

Let consider the mass spring damper system :





From the Newton's second law :

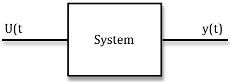
$$M\ddot{x} = -kx - f\dot{x} + F$$

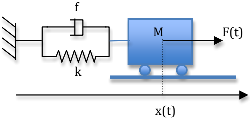
which is usually treated using the canonical state space representation :

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -f \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F$$

A simple example ...

Let consider the mass spring damper system :





From the Newton's second law :

$$M\ddot{x} = -kx - f\dot{x} + F$$

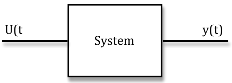
An alternative representation consist in choosing the energy variables (extensives variables) as state variables *i.e* $(x, p = M\dot{x})$

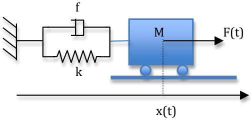
$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix}}_{J-R} \underbrace{\begin{pmatrix} kx \\ \dot{x} \end{pmatrix}}_{\partial_x H} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} F$$

with $H(x, p) = kx^2 + \frac{1}{M}p^2$

A simple example ...

Let consider the mass spring damper system :





From the Newton's second law :

$$M\ddot{x} = -kx - f\dot{x} + F$$

Defining y s.t. :

$$\left\{ \begin{array}{lcl} \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} & = & \begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix} \begin{pmatrix} \partial_x H(x,p) \\ \partial_p H(x,p) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F \\ y & = & \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x H(x,p) \\ \partial_p H(x,p) \end{pmatrix} \end{array} \right.$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}^T \frac{dx}{dt} = \frac{\partial H}{\partial x}^T (J - R) \frac{\partial H}{\partial x} + \frac{\partial H}{\partial x}^T Bu \leq y^T u$$

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Back to the modeling

The previous model can be written from the interconnection of a subset of basic mechanical elements :

- A moving inertia.
- A spring.
- A damper.
- A source and some interconnection relations.

Structured modeling

Each element is characterized by a set of power conjugated variables, the flow variables and the effort variables (intensive variables). The state variable is derive from the time integration of the flow variables (extensive variables). When the component is purely dissipative there is no associated state variable.

Moving inertia

Set of power conjugated variables :

- Flow variable : Force

$$\frac{dp}{dt} = F$$

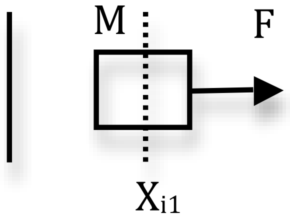
- Effort variable : velocity

$$v_i(p) = \frac{1}{m}p$$

State variable and energy

- Extensive variable : kinetic momentum p
- Energy

$$E(p) = \frac{1}{2}\frac{p^2}{m}$$



Spring

Set of power conjugated variables :

- Flow variable : Velocity

$$\frac{dx}{dt} = v_s$$

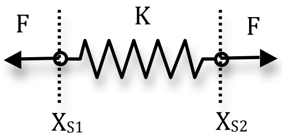
- Effort variable : Force

$$F(x) = kx$$

State variable and energy

- Extensive variable : position x
- Energy

$$E(x) = \frac{1}{2}kx^2$$



Damper

Set of power conjugated variables :

- Flow variable : Velocity

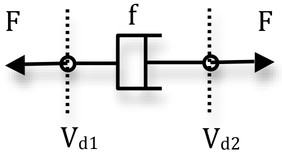
$$v_d$$

- Effort variable : Force

$$F = kv_d$$

Dissipated (co)energy :

$$D(v) = kv_d^2$$



Transformers and sources

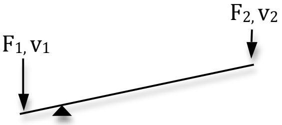
Power preserving transformations :

- Relation between velocities

$$v_2 = nv_1$$

- Relation between forces

$$F_1 = nF_2$$



There exist different kind of sources

- Velocity sources

$$v(t) = v_s(t)$$

- Forces sources,

$$F(t) = F_s(t)$$

Interconnection

When two or more mechanical subsystems are interconnected one can write at the interconnection point :

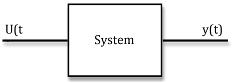
- Equality of the velocities,

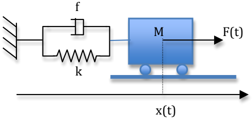
$$v_d = v_s = v_i = v$$

- Forces balance,

$$F_i + F_s + F_d = F$$

Back to the example





- Equality of the velocities,

$$v_d = v_s = v_i = v$$

- Forces balance,

$$F_i + F_s + F_d = F$$

States variables : $(x \ p)^T$

$$\frac{dx}{dt} = v_s = v$$

$$\frac{dp}{dt} = F - F_s - F_d = F - kx - fv$$

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Electric circuits

Coupling between electric fields and magnetic fields

- Capacitors.
- Inductors.
- Resistors.
- Transformers and sources.
- Interconnection relations.

Capacitors

Set of power conjugated variables :

- Flow variable : Current

$$\frac{dq}{dt} = i$$

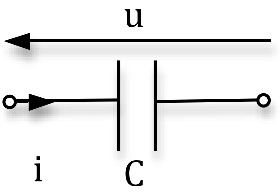
- Effort variable : Voltage

$$V(q) = \frac{1}{C}q$$

State variable and energy

- Extensive variable : charge q
- Energy

$$E(x) = \frac{1}{2}\frac{1}{C}q^2$$



Inductors

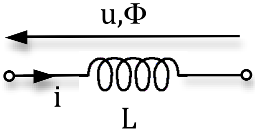
Set of power conjugated variables :

- Flow variable : Voltage

$$\frac{d\phi}{dt} = u$$

- Effort variable : Current

$$I(\phi) = \frac{1}{L}\phi$$



State variable and energy

- Extensive variable : Flux-linkage ϕ
- Energy :

$$E(x) = \frac{1}{2}\frac{1}{L}\phi^2$$

Resistors

Set of power conjugated variables :

- Flow variable : Current

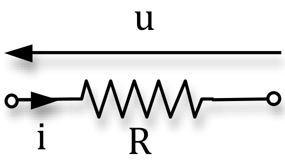
$$i_r$$

- Effort variable : Voltage

$$u = Ri_r$$

Dissipated (co)energy :

$$D(i_r) = Ri_r^2$$



Interconnection

- Kirchhoff's Current Law (KCL) :
$$\sum_k i_k = 0$$
for each node.
- Kirchhoff's Voltage Law (KVL) :
$$\sum_k u_k = 0$$

for each loop.

Transformer and sources

Transformer

- Relationships :

$$u_1 = nu_2,\ i_1 = \frac{i_2}{n}$$

- Power preserving reprsentation ($i_1 u_1 = i_2 u_2$)

Sources

- Voltage source

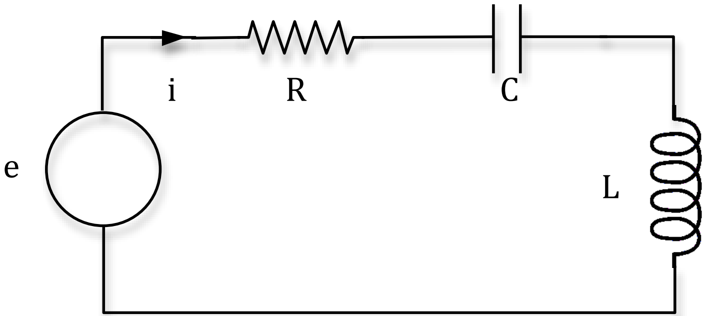
$$v = v_s$$

- Current source

$$i = i_s$$

Example

Linear RLC Circuit



Example

The system is made up with four elements :

- Voltage source : $u = e, i$
- Capacitor (Q_c, i_c, u_c)
- Inductor (Φ_L, v_L, i_L)
- Resistor (i_R, u_R)

The interconnection is given by :

$$u = u_R + u_L + u_C, \quad i = i_R = i_L = i_C$$

Dynamic equations :

$$\frac{d\Phi_L}{dt} = u_L, \quad \frac{dQ_c}{dt} = i_c$$

Example

Port Hamiltonian formulation. The dynamics is given by

$$\frac{d}{dt}\begin{pmatrix} Q_c \\ \Phi_L \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -R \end{pmatrix} \begin{pmatrix} \frac{1}{C} Q_c \\ \frac{1}{L}\Phi_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

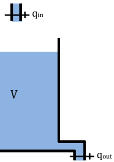
with output mapping :

$$i = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{C} Q_c \\ \frac{1}{L}\Phi_L \end{pmatrix}$$

The energy is given by $E = \frac{1}{2}\left(\frac{1}{C}Q_c^2 + \frac{1}{L}\Phi_L^2\right)$ with balance

$$\frac{dE}{dt} = u^T i - Ri^2$$

Hydraulic systems



- State variable : Volume V
- Flow variable : Volume flow
- Effort variable : Pressure $P = \frac{\rho}{A} V g$

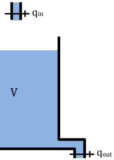
The dynamic equation if given by :

$$\frac{dV}{dt} = q_{in} - q_{out} \quad \text{where} \quad q_{out} = C\sqrt{h}$$

The energy is defined by :

$$E(V) = \int_V P dV = \frac{1}{2} \frac{\rho}{A} V^2 g$$

Hydraulic systems



Then

$$\frac{dV}{dt} = -\frac{C}{\sqrt{h}}\frac{\partial E}{\partial V} + q_{in}$$

and

$$P = \frac{\partial E}{\partial V} = \rho g h$$

with

$$E(V) = \int_V P dV = \frac{1}{2}\frac{\rho}{A}V^2 g$$

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Port based modeling of physical systems

Port Hamiltonian formulation

The idea is to generalize what has been proposed for mechanical and electrical systems to other class of systems.

Why ?

- We have pointed out some common properties : storage, dissipation and transformation.
- Engineering systems are a combination of subsystems related to possible different physical domains and interconnection has to be consistent. See for example Adsorption processes.
- Decomposition in basic elements helps in modeling of complex dynamic systems (coming from different areas).
- Modeling is attached to the notion of graph.

Port based modeling of physical systems

Much more fundamental reasons :

- Central role of the energycan be used for control purposes. Lyapunov based approaches.
- More information are taken into account in the model through symmetries.
- The model is a knowledge based model that takes the non linearities and the distributed aspects into account.

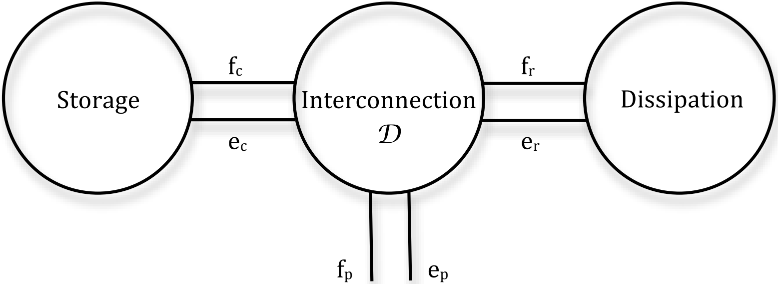
Generalized Bond Graph

Decomposition in basic elements is linked to Generalized Bond Graph (Paytner, Breedveld) :

- Systems are decomposed in elements with specific energetic behavior : storage, dissipation and transformation.
- Each element is characterized by a pair of power conjugated variables : the flow variables $f \in \mathcal{F}$ and the effort variables $e \in \mathcal{E}$. The associated power port is given by :

$$P = f^T e$$

Port based modeling



$\mathcal{F} = \mathcal{F}_c \times \mathcal{F}_R \times \mathcal{F}_p$ and $\mathcal{E} = \mathcal{E}_c \times \mathcal{E}_R \times \mathcal{E}_p$

Dynamic relations : storage element

In case of storage elements :

- The state variable x is the extensive variable of Thermodynamics. It is linked to the flow variables through the balance equation :

$$\frac{dx}{dt} = -f_c$$

- The effort variable is linked to the energy variable through the relation :

$$e_c = e_c(x) = \frac{dE}{dx}$$

- The Energy balance is given by

$$\frac{dE}{dt} = \left(\frac{dE}{dx}\right)^T \left(\frac{dx}{dt}\right) = e_c^T f_c$$

Dissipation

In the case of dissipation :

$$e_r = -e(f); \quad f = f_r$$

or

$$f_r = -f(e); \quad e = e_r$$

Such that

$$e^T f(e) \geq 0, \quad e(f)^T f \geq 0$$

Examples :

$$u = Ri, \; D = u^T i = Ri^2$$

$$F = f\dot{x}, \; D = \dot{x}F = f\dot{x}^2$$

Then

$$e_R^T f_R \leq 0$$

Interconnexion

- 1 Junction (flow junction) :
  - Equality of effort variables
  - Balance on the flow variables

  Example : Kirchhoff's voltage law
- 0 Junction (flow junction) :
  - Equality of flow variables
  - Balance on the effort variables

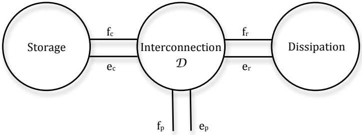
  Example : Kirchhoff's current law
- Ideal transformer "TF" :

$$\begin{pmatrix} e_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ e_2 \end{pmatrix}, \quad e_1^T f_1 = e_2^T f_2$$

- Ideal gyrator "TF" :

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad e_1^T f_1 = e_2^T f_2$$

Interconnection structure and power balance



The power balance is given by :

$$e_c^T f_c + e_R^T f_R^T + e_p^T f_p = 0$$

And

$$\frac{dE}{dt} = \left(\frac{dE}{dx}\right)^T \frac{dx}{dt} = -e_c^T f_c = e_R^T f_R^T + e_p^T f_p$$

and then

$$E(t) = E(0) + \underbrace{\int_t e_R^T f_R^T dt}_{dissipated\ energy} + \underbrace{\int_t e_p^T f_p dt}_{exchanged\ energy}$$

Physical domain	flow $f \in \mathcal{F}$	effort $e \in \mathcal{E}$	state
potential translation	velocity	force	displacement
kinetic translation	force	velocity	momentum
potential rotation	angular velocity	torque	angle
kinetic rotation	torque	angular velocity	angular momentum
electric	current	voltage	charge
magnetic	voltage	current	flux linkage
potential hydraulic	volume flow	pressure	volume
kinetic hydraulic	pressure	volume flow	flow momentum
chemical	molar flow	chemical potential	number of moles
thermal	entropy flow	temperature	entropy

Back to the energy

Well known subsystems with linear closure relations

- Potential energy stored in a spring : $e = F(x)$

$$E(x) = \int_x Kxdx = \frac{1}{2}Kx^2$$

- Kinetic energy oh a mass : $e = v(p) = \frac{p}{M}$

$$E(p) = \int_p \frac{p}{M}dp = \frac{1}{2}\frac{p^2}{M}$$

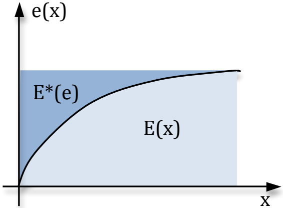
but it can be derived in case of non linear closure relations

- Potential energy stored in a non linear spring : $e = K(x) = K_0 + K_1 X + K_2 X^2$

$$E(x) = \int_x K(x)dx = K_0 x + \frac{1}{2}K_1 x^2 + \frac{1}{3}K_2 x^3$$

Energy and co energy

The energy is expressed in term of the energy variables (extensive variables of Thermodynamics), i.e. $E(x) = \int_x e(x)dx$ where $e(x)$ is the co-energy variables (intensive variables of Thermodynamics). Graphically $E(x)$ is the surface under the curve $e(x)$.



if $e = e(x)$ is reversible,

$$E^*(e) = \int_e x(e)de$$

is the co energy of the system. It is the Legendre transform of the energy *i.e.* $E(e) = xe - E(x)$ with $e = \frac{dE}{dx}$. In the linear case $E(x) = E^*(e)$

Energy and co energy

In the case of moving inertia :

- Effort variable $e = v$
- State variable p

Or $p = Mv$ then

$$E(p) = \int_{v} p(v)dv = \frac{1}{2}Mv^2$$

In this case the Legendre transform of the energy is given by

$$E^*(v) = pv - E(p) = pv - \frac{1}{2}Mv^2 = \frac{1}{2}Mv^2 = E(p)$$

Energy and co energy

Let's now consider a non quadratic energy function :

$$E(x) = \frac{1}{6}x^6$$

with $e(x) = x^5$. Then the co energy reads

$$E^*(e) = \left(xe - \frac{1}{6}x^6\right)\Big|_{x=e^{\frac{1}{5}}} = e^{\frac{6}{5}} - \frac{1}{6}e^{\frac{6}{5}} = \frac{5}{6}e^{\frac{6}{5}}$$

Then

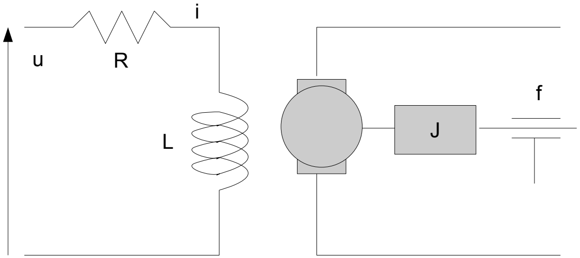
$$E^*(e) \neq E(x)$$

Furthermore

$$\dot{E}^*(e) = \left(\frac{dE}{de}\right)^T \frac{de}{dt} = x\dot{e} \neq e^T f$$

Exercice

Propose a port Hamiltonian model of the DC motor



Dirac structures and Port Hamiltonian systems

To summarize, the overall system is defined from pairs of flow variables, effort variables and state variables x. They are made up with :

- Energy storing elements :

$$f_c = -\frac{dx}{dt}, \ e_c = \frac{\partial E}{\partial x}$$

- Power dissipating elements

$$R(f_R, e_R) = 0, \ e_R^T f_R \geq 0$$

- Power preserving transformers, gyrators.
- Power preserving junctions.

$\Rightarrow$ Interconnexion structure = Dirac structure

Geometric structure

Dirac structure

A constant Dirac structure on a finite dimensional space $\mathcal{V}$ is subspace

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that

1. $e^T f = 0$ for all $(f, e) \in \mathcal{D}$
2. $dim\mathcal{D} = dim\mathcal{V}$

For any skew-symmetric map $J : \mathcal{V}^* \to \mathcal{V}$ its graph $\{(f, e) \in \mathcal{V} \times \mathcal{V}^* | f = Je\}$ is a Dirac structure.

Geometric structure

Dirac structure 2

A constant Dirac structure on a finite dimensional space $\mathcal{V}$ is subspace

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that

$$\mathcal{D} = \mathcal{D}^{\perp}$$

where $\perp$ denotes orthogonal complement with respect to the bilinear form $\ll, \gg$ defined as :

$$\ll (f_1, e_1), (f_2, e_2) \gg = \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle$$

with $\langle e|f \rangle = e^T f$ the natural power product.

Geometric structure

Port Hamiltonian system

The dynamical system defined by DAEs such that :

$$(f_c, e_c, f_p, e_P) \in \mathcal{D}, \ t \in \mathbb{R}$$

with $f_c = \frac{\partial E}{\partial} e_c = \frac{\partial E}{\partial}$ is called port Hamiltonian system.