Introduction to modeling of Port Hamiltonian Systems

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Outline

1. Dynamic systems modeling
2. Port Hamiltonian framework
3. Back to modeling
4. Port Hamiltonian modeling
In this course:
Non linear, multi physic, multi scale, distributed parameters systems.
Example 1: inverted pendulum system

Example: Segway system
Example 1: inverted pendulum system

Non linear mechanical system:
- Two natural equilibria.
- Control: insure $\Theta = 0$
Example 2: Nanotweezer for DNA manipulation

Introduction

Biocharacterizations on DNA

Control of tweezers

Conclusions

Single molecule techniques

Silicon nanotweezers for DNA experiments

Design of the silicon nanotweezers

[Image of a diagram showing the components of a nanotweezer, including electrodes for DNA trapping, capacitive sensor, differential capacitive sensor, tips for molecule manipulation, actuation voltage, and comb drive actuator.]

[Image showing a nanotweezer in action, manipulating a DNA molecule.]
Example 2: Nanotweezer for DNA manipulation
Example 3: Ionic Polymer Metal Composite

- Electromechanical system.
- 3 scales: Polymer-electrode interface, diffusion in the polymer, beam bending.
Example 4 : Active skin for vibro-acoustic control

Enfin la troisième équation est une relation thermodynamique qui relie la pression à la masse volumique:
\[
\frac{dp}{d\rho} = \frac{1}{\rho_s}\chi_s \quad (3)
\]

\(\chi_s\) : coëfficient de compressibilité adiabatique((Pa−1))

Le coëfficient \(\chi_s\) est la compressibilité adiabatique, c'est-à-dire que l'on s'interesse à la variation de pression avec le changement de température.

2 Energie acoustique

Les paramètres des équations de propagation de l'onde acoustique dans un tube cf. figure 1, vont nous permettre d'exprimer la densité d'énergie acoustique qui dépend à la fois du carré de la pression (énergie potentielle) et du carré de la vitesse (énergie cinétique).

\[
\rho_0 \chi_s \frac{d^2 p}{dt^2} - \Delta p = 0
\]

\(\Delta p\) est le changement de pression.

Equation de propagation faisant intervenir la vitesse vibratoire [3]:
\[
\rho_0 \chi_s \frac{d^2 v}{dt^2} - \Delta v = 0
\]

L'équation de propagation (en pression) en coordonnées cartésiennes (2D):
\[
\rho_0 \chi_s \frac{d^2 p}{dt^2} - \frac{d^2 p}{dx^2} - \frac{d^2 p}{dy^2} = 0
\]

2-D case :
- 2-D wave equation
- Non linear finite dimensional system : loudspeakers/microphones
- Power preserving interconnection
Example 5: Adsorption process

- Multiscale heterogeneous system.
- Dynamic behavior driven by irreversible thermodynamic laws
Example 5 : Adsorption process

- Multiscale heterogeneous system.
- Considered phenomena :
  - Fluid scale : convection, dispersion.
  - Pellet scale : diffusion (Stephan-Maxwell).
  - Microscopic scale : Knudsen law.
Toward more complex systems ...

Tokamak nuclear reactor
Models and Complexity

- A model is always an approximation of reality.
- A model depends on the problem context.
- A model has to be tractable.

### Purpose

Derive a mathematical model based on Physics useful for:
- Simulation (model reduction)
- Analysis
- Control design
Models and Complexity (illustration)

- Extra granular phase
  - L \( \frac{1}{4} \) 25 cm
  - \( R_{\text{int}} \) \( \frac{1}{4} 1\) cm

- Macropore scale
  - \( R_p \) \( \frac{1}{4} 1.24 \) mm

- Micropore scale
  - \( R_c \) \( \frac{1}{4} 1 \) µm

- Bidisperse pellet
- Microporous crystal

Finite difference with \( N=10 \)

Structural method \( N=10 \)

Figure 4: Principle of the spatial discretization
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Port Hamiltonian framework

Philosophy
- Use of power conjugated variables names "flows" and "efforts" variables.
- Associated with a powerful graphical tool: the Bond Graphs.

Characteristics
- Formalism coming from differential geometry (free of coordinates, useful for model reduction).
- Suitable for functional analysis (finite and infinite dimension) and system control theory.
Port Hamiltonian framework

Port Hamiltonian systems

Class of non linear dynamic systems derived from an extension to open physical systems (1992) of Hamiltonian and Gradient systems. This class has been generalized (2001) to distributed parameter systems.

\[
\begin{align*}
\dot{x}(t) & : \quad \dot{x} = (J(x) - R(x)) \frac{\partial H(x)}{\partial x} + B(x)u \\
y(t) & = B(x)^T \frac{\partial H(x)}{\partial x}
\end{align*}
\]

\[
\begin{array}{ll}
x(t, \theta) : & \dot{x} = (J(x) - R(x)) \frac{\delta H(x)}{\delta x} \\
& \begin{pmatrix} f_{\theta} \\ e_{\theta} \end{pmatrix} = \frac{\delta H(x)}{\delta x} |_{\theta}
\end{array}
\]

- Central role of the energy.
- Additional information coming from the geometric structure.
- Multi-physic framework.
A simple example ...

Let consider the mass spring damper system:

From the Newton’s second law:

\[ M \ddot{x} = -kx - f \dot{x} + F \]

which is usually treated using the canonical state space representation:

\[
\begin{pmatrix}
\dot{x} \\
\ddot{x}
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-\frac{k}{M} & -f
\end{pmatrix}
\begin{pmatrix}
x \\
\dot{x}
\end{pmatrix}
+ \begin{pmatrix}
0 \\
1
\end{pmatrix} F
\]
A simple example ...

Let consider the mass spring damper system:

From the Newton’s second law:

\[ M\ddot{x} = -kx - f\dot{x} + F \]

An alternative representation consist in choosing the energy variables (extensives variables) as state variables i.e \((x, p = M\dot{x})\)

\[
\begin{pmatrix}
\dot{x} \\
\dot{p}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 \\
-1 & -f
\end{pmatrix}
\begin{pmatrix}
kx \\
\dot{x}
\end{pmatrix}
+ 
\begin{pmatrix}
0 \\
1
\end{pmatrix}
F
\]

with \(H(x, p) = kx^2 + \frac{1}{M}p^2\)
A simple example ...

Let consider the mass spring damper system:

From the Newton’s second law:

\[ M\ddot{x} = -kx - f\dot{x} + F \]

Defining \( y \) s.t.:

\[
\begin{cases}
\begin{pmatrix} \dot{x} \\ \dot{\rho} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix} \begin{pmatrix} \partial_x H(x, \rho) \\ \partial_\rho H(x, \rho) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F \\
y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x H(x, \rho) \\ \partial_\rho H(x, \rho) \end{pmatrix}
\end{cases}
\]

\[
\frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} = \frac{\partial H}{\partial x} (J - R) \frac{\partial H}{\partial x} + \frac{\partial H}{\partial x} Bu \leq y^T u
\]
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4. Port Hamiltonian modeling
Back to the modeling

The previous model can be written from the interconnection of a subset of basic mechanical elements:

- A moving inertia.
- A spring.
- A damper.
- A source and some interconnection relations.

Structured modeling

Each element is characterized by a set of power conjugated variables, the flow variables and the effort variables (intensive variables). The state variable is derived from the time integration of the flow variables (extensive variables). When the component is purely dissipative there is no associated state variable.
Moving inertia

Set of power conjugated variables:

- Flow variable: Force
  \[
  \frac{dp}{dt} = F
  \]
- Effort variable: velocity
  \[
  v_i(p) = \frac{1}{m} p
  \]

State variable and energy:

- Extensive variable: kinetic momentum \( p \)
- Energy
  \[
  E(p) = \frac{1}{2} \frac{p^2}{m}
  \]
Spring

Set of power conjugated variables:
- Flow variable: Velocity
  \[ \frac{dx}{dt} = v_s \]
- Effort variable: Force
  \[ F(x) = kx \]

State variable and energy
- Extensive variable: position \( x \)
- Energy
  \[ E(x) = \frac{1}{2} kx^2 \]
Damper

Set of power conjugated variables:

- Flow variable: Velocity
  \[ v_d \]
- Effort variable: Force
  \[ F = kv_d \]

Dissipated (co)energy:

\[ D(v) = kv_d^2 \]
Transformers and sources

Power preserving transformations:
- Relation between velocities
  \[ v_2 = n v_1 \]
- Relation between forces
  \[ F_1 = n F_2 \]

There exist different kind of sources
- Velocity sources
  \[ v(t) = v_s(t) \]
- Forces sources,
  \[ F(t) = F_s(t) \]
Interconnection

When two or more mechanical subsystems are interconnected one can write at the interconnection point:

- Equality of the velocities,
  \[ v_d = v_s = v_i = v \]

- Forces balance,
  \[ F_i + F_s + F_d = F \]
Equality of the velocities,

\[ v_d = v_s = v_i = v \]

Forces balance,

\[ F_i + F_s + F_d = F \]

States variables : \((x \ p)^T\)

\[
\frac{dx}{dt} = v_s = v
\]

\[
\frac{dp}{dt} = F - F_s - F_d = F - kx - fv
\]
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Electric circuits

Coupling between electric fields and magnetic fields

- Capacitors.
- Inductors.
- Resistors.
- Transformers and sources.
- Interconnection relations.
Capacitors

Set of power conjugated variables:

- Flow variable: Current
  \[ \frac{dq}{dt} = i \]

- Effort variable: Voltage
  \[ V(q) = \frac{1}{C} q \]

State variable and energy

- Extensive variable: charge \( q \)
- Energy
  \[ E(x) = \frac{1}{2} \frac{1}{C} q^2 \]
Inductors

Set of power conjugated variables:
- Flow variable: Voltage
  \[
  \frac{d\phi}{dt} = u
  \]
- Effort variable: Current
  \[
  I(\phi) = \frac{1}{L} \phi
  \]

State variable and energy
- Extensive variable: Flux-linkage \( \phi \)
- Energy:
  \[
  E(x) = \frac{1}{2} \frac{1}{L} \phi^2
  \]
Resistors

Set of power conjugated variables:
- Flow variable: Current
  \[ i_r \]
- Effort variable: Voltage
  \[ u = R i_r \]

Dissipated (co)energy:
\[ D(i_r) = R i_r^2 \]
Interconnection

- Kirchhoff’s Current Law (KCL) :\[ \sum_{k} i_k = 0 \]
  for each node.
- Kirchhoff’s Voltage Law (KVL) :\[ \sum_{k} u_k = 0 \]
  for each loop.
Transformer and sources

Transformer
- Relationships:
  \[ u_1 = n u_2, \quad i_1 = \frac{i_2}{n} \]
- Power preserving representation \((i_1 u_1 = i_2 u_2)\)

Sources
- Voltage source
  \[ v = v_s \]
- Current source
  \[ i = i_s \]
Example

Linear RLC Circuit
Example

The system is made up with four elements:

- Voltage source: \( u = e, i \)
- Capacitor: \( Q_c, i_c, u_c \)
- Inductor: \( \Phi_L, v_L, i_L \)
- Resistor: \( i_R, u_R \)

The interconnection is given by:

\[
    u = u_R + u_L + u_C, \quad i = i_R = i_L = i_C
\]

Dynamic equations:

\[
    \frac{d\Phi_L}{dt} = u_L, \quad \frac{dQ_c}{dt} = i_c
\]
Example

Port Hamiltonian formulation. The dynamics is given by

\[
\frac{d}{dt} \begin{pmatrix} Q_c \\ \Phi_L \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -R \end{pmatrix} \begin{pmatrix} \frac{1}{C} Q_c \\ \frac{1}{L} \Phi_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u
\]

with output mapping:

\[
i = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{C} Q_c \\ \frac{1}{L} \Phi_L \end{pmatrix}
\]

The energy is given by \( E = \frac{1}{2} \left( \frac{1}{C} Q_c^2 + \frac{1}{L} \Phi_L^2 \right) \) with balance

\[
\frac{dE}{dt} = u^T i - R i^2
\]
Hydraulic systems

- State variable: Volume $V$
- Flow variable: Volume flow
- Effort variable: Pressure $P = \frac{\rho}{A} Vg$

The dynamic equation is given by:

$$\frac{dV}{dt} = q_{in} - q_{out} \quad \text{where} \quad q_{out} = C\sqrt{h}$$

The energy is defined by:

$$E(V) = \int_V PdV = \frac{1}{2} \frac{\rho}{A} V^2 g$$
Hydraulic systems

Then

\[
\frac{dV}{dt} = -\frac{C}{\sqrt{h}} \frac{\partial E}{\partial V} + q_{in}
\]

and

\[
P = \frac{\partial E}{\partial V} = \rho gh
\]

with

\[
E(V) = \int_V PdV = \frac{1}{2} \frac{\rho}{A} V^2 g
\]
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Port based modeling of physical systems

Port Hamiltonian formulation

The idea is to generalize what has been proposed for mechanical and electrical systems to other class of systems.

Why?

- We have pointed out some common properties: storage, dissipation and transformation.
- Engineering systems are a combination of subsystems related to possible different physical domains and interconnection has to be consistent. See for example Adsorption processes.
- Decomposition in basic elements helps in modeling of complex dynamic systems (coming from different areas).
- Modeling is attached to the notion of graph.
Port based modeling of physical systems

Much more fundamental reasons:

- Central role of the energy can be used for control purposes. Lyapunov based approaches.

- More information are taken into account in the model through symmetries.

- The model is a knowledge based model that takes the non-linearities and the distributed aspects into account.
Generalized Bond Graph

Decomposition in basic elements is linked to Generalized Bond Graph (Paytner, Breedveld):

- Systems are decomposed in elements with specific energetic behavior: storage, dissipation and transformation.

- Each element is characterized by a pair of power conjugated variables: the flow variables \( f \in \mathcal{F} \) and the effort variables \( e \in \mathcal{E} \). The associated power port is given by:

\[
P = f^T e
\]
Port based modeling

\[ F = F_c \times F_R \times F_p \text{ and } E = E_c \times E_R \times E_p \]
In case of storage elements:

- The state variable $x$ is the extensive variable of Thermodynamics. It is linked to the flow variables through the balance equation:

\[
\frac{dx}{dt} = -f_c
\]

- The effort variable is linked to the energy variable through the relation:

\[
e_c = e_c(x) = \frac{dE}{dx}
\]

- The Energy balance is given by

\[
\frac{dE}{dt} = \left( \frac{dE}{dx} \right)^T \left( \frac{dx}{dt} \right) = e_c^T f_c
\]
Dissipation

In the case of dissipation:

\[ e_r = -e(f); \quad f = f_r \]

or

\[ f_r = -f(e); \quad e = e_r \]

Such that

\[ e^T f(e) \geq 0, \quad e(f)^T f \geq 0 \]

Examples:

\[ u = Ri, \quad D = u^T i = Ri^2 \]
\[ F = f\dot{x}, \quad D = \dot{x}F = f\dot{x}^2 \]

Then

\[ e_R^T f_R \leq 0 \]
Interconnexion

• 1 Junction (flow junction) :
  • Equality of effort variables
  • Balance on the flow variables
  Example : Kirchhoff’s voltage law

• 0 Junction (flow junction) :
  • Equality of flow variables
  • Balance on the effort variables
  Example : Kirchhoff’s current law

• Ideal transformer "TF" :

\[
\begin{pmatrix}
e_1 \\
f_2 \\
\end{pmatrix} = \begin{pmatrix}
0 & n \\
n & 0 \\
\end{pmatrix} \begin{pmatrix}
f_1 \\
e_2 \\
\end{pmatrix}, \quad e_1^T f_1 = e_2^T f_2
\]

• Ideal gyrator "TF" :

\[
\begin{pmatrix}
e_1 \\
e_2 \\
\end{pmatrix} = \begin{pmatrix}
0 & n \\
n & 0 \\
\end{pmatrix} \begin{pmatrix}
f_1 \\
f_2 \\
\end{pmatrix}, \quad e_1^T f_1 = e_2^T f_2
\]
Interconnection structure and power balance

The power balance is given by:

\[ e_c^T f_c + e_R^T f_R + e_p^T f_p = 0 \]

And

\[ \frac{dE}{dt} = \left( \frac{dE}{dx} \right)^T \frac{dx}{dt} = -e_c^T f_c = e_R^T f_R + e_p^T f_p \]

and then

\[ E(t) = E(0) + \int_t e_R^T f_R dt + \int_t e_p^T f_p dt \]

\[ \text{dissipated energy} \quad \text{exchanged energy} \]
<table>
<thead>
<tr>
<th>Physical domain</th>
<th>flow $f \in F$</th>
<th>effort $e \in E$</th>
<th>state</th>
</tr>
</thead>
<tbody>
<tr>
<td>potential translation</td>
<td>velocity</td>
<td>force</td>
<td>displacement</td>
</tr>
<tr>
<td>kinetic translation</td>
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<td>velocity</td>
<td>momentum</td>
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<td>entropy flow</td>
<td>temperature</td>
<td>entropy</td>
</tr>
</tbody>
</table>
Well known subsystems with linear closure relations

- Potential energy stored in a spring: \( e = F(x) \)

\[
E(x) = \int_x Kxdx = \frac{1}{2} Kx^2
\]

- Kinetic energy oh a mass: \( e = v(p) = \frac{p}{M} \)

\[
E(p) = \int_p \frac{p}{M} dp = \frac{1}{2} \frac{p^2}{M}
\]

but it can be derived in case of non linear closure relations

- Potential energy stored in a non linear spring: \( e = K(x) = K_0 + K_1 X + K_2 X^2 \)

\[
E(x) = \int_x K(x) dx = K_0 x + \frac{1}{2} K_1 x^2 + \frac{1}{3} K_2 x^3
\]
Energy and co energy

The energy is expressed in terms of the energy variables (extensive variables of Thermodynamics), i.e. $E(x) = \int x e(x) dx$ where $e(x)$ is the co-energy variables (intensive variables of Thermodynamics). Graphically $E(x)$ is the surface under the curve $e(x)$.

If $e = e(x)$ is reversible, $E^*(e) = \int_e x(e) de$ is the co-energy of the system. It is the Legendre transform of the energy i.e. $E(e) = xe - E(x)$ with $e = \frac{dE}{dx}$. In the linear case $E(x) = E^*(e)$.
Energy and co energy

In the case of moving inertia:

- Effort variable $e = v$
- State variable $p$

Or $p = Mv$ then

$$E(p) = \int_v p(v) dv = \frac{1}{2}Mv^2$$

In this case the Legendre transform of the energy is given by

$$E^*(v) = pv - E(p) = pv - \frac{1}{2}Mv^2 = \frac{1}{2}Mv^2 = E(p)$$
Energy and co energy

Let's now consider a non quadratic energy function:

\[ E(x) = \frac{1}{6} x^6 \]

with \( e(x) = x^5 \). Then the co energy reads

\[ E^*(e) = \left( xe - \frac{1}{6} x^6 \right) \bigg|_{x=e^{1/5}} = e^{6/5} - \frac{1}{6} e^{6/5} = \frac{5}{6} e^{6/5} \]

Then

\[ E^*(e) \neq E(x) \]

Furthermore

\[ \dot{E}^*(e) = \left( \frac{dE}{de} \right)^T \frac{de}{dt} = x \dot{e} \neq e^T f \]
Exercice

Propose a port Hamiltonian model of the DC motor

![Diagram of a DC motor with terminals labeled u, R, L, i, and f.](image)
Dirac structures and Port Hamiltonian systems

To summarize, the overall system is defined from pairs of flow variables, effort variables and state variables $x$. They are made up with:

- **Energy storing elements**:
  
  $$f_c = -\frac{dx}{dt}, \quad e_c = \frac{\partial E}{\partial x}$$

- **Power dissipating elements**
  
  $$R(f_R, e_R) = 0, \quad e^T_R f_R \geq 0$$

- **Power preserving transformers, gyrators.**
- **Power preserving junctions.**

$\Rightarrow$ Interconnexion structure = Dirac structure
Geometric structure

**Dirac structure**

A constant Dirac structure on a finite dimensional space $\mathcal{V}$ is subspace

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that

1. $e^T f = 0$ for all $(f, e) \in \mathcal{D}$
2. $\dim \mathcal{D} = \dim \mathcal{V}$

For any skew-symmetric map $J : \mathcal{V}^* \to \mathcal{V}$ its graph $\{(f, e) \in \mathcal{V} \times \mathcal{V}^* | f = Je\}$ is a Dirac structure.
A constant Dirac structure on a finite dimensional space $\mathcal{V}$ is subspace

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that

$$\mathcal{D} = \mathcal{D}^\perp$$

where $\perp$ denotes orthogonal complement with respect to the bilinear form $\ll, \gg$ defined as :

$$\ll (f_1, e_1), (f_2, e_2) \gg = \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle$$

with $\langle e | f \rangle = e^T f$ the natural power product.
Geometric structure

Port Hamiltonian system

The dynamical system defined by DAEs such that:

\[(f_c, e_c, f_p, e_P) \in \mathcal{D}, \ t \in \mathbb{R}\]

with \(f_c = \frac{\partial E}{\partial e_c} = \frac{\partial E}{\partial o}\) is called port Hamiltonian system.